

18. Barotropic instability

Consider an inviscid barotropic flow governed by the barotropic vorticity equation

$$\frac{d\eta}{dt} = 0, \quad (18.1)$$

where

$$\eta = \nabla^2 \psi \quad (18.2)$$

and ψ is the streamfunction. There exists a class of exact solutions of (18.1) characterized by

$$\begin{aligned} \psi &= \psi(y), \\ \eta &= \frac{d^2\psi}{dy^2}. \end{aligned}$$

These are just zonal flows that vary meridionally. We have seen that flows with cross-stream variations of η can support Rossby waves. Now consider a class of jet-like flows that look like the example shown in Figure 18.1, where there is an extremum in the vorticity. In the example given, there is a maximum of vorticity in the center of the domain, with westward flow to the north and eastward flow to the south. The meridional gradient of vorticity is negative to the north of the vorticity maximum, and positive to the south, so that barotropic Rossby waves propagate eastward, relative to the flow, to the north; and westward, relative to the flow, to the south.

Now consider perturbing the flow in Figure 18.1 in the manner shown in Figure 18.2:

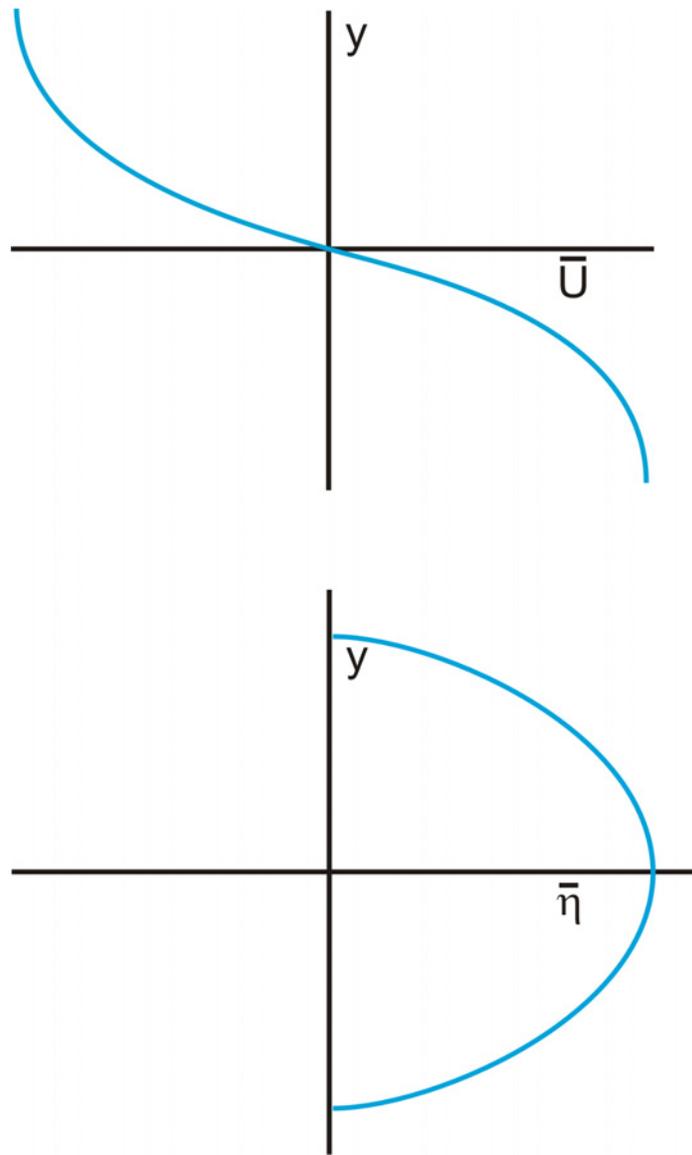


Figure 18.1

This particular deformation of the background vorticity contours produces positive perturbations of vorticity along a southwest-northeast axis. The sense of the perturbation flow associated with these vorticity anomalies is also illustrated in Figure 18.2. Note the follow major points:

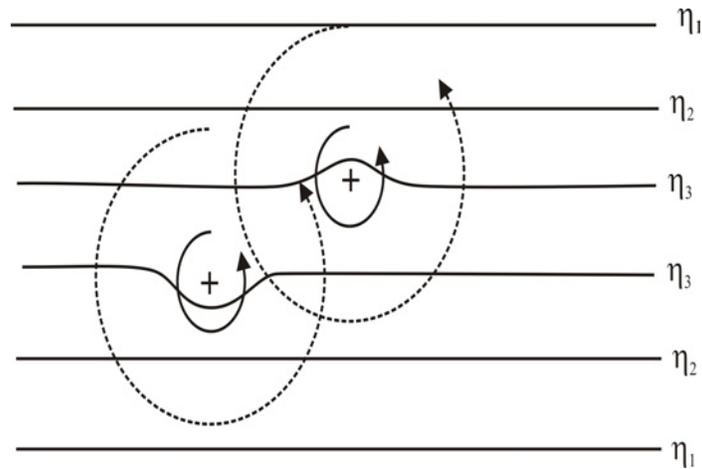


Figure 18.2

1. The local advection of the background gradient of vorticity by the flow associated with the northern perturbation is such as to propagate it eastward *relative to the background flow*, which is toward the west. Thus the intrinsic propagation is opposite to the background flow.
2. The local advection of the background vorticity by the flow associated with the southern vorticity anomaly is such as to propagate the anomaly westward *relative to the background flow*, which is toward the east. Again, the intrinsic propagation is opposite to the background flow.
3. In the vicinity of the *southern* vorticity anomaly, the advection of the background vorticity by the flow associated with the *northern* vorticity anomaly is such as to amplify the southern vorticity anomaly. The same goes for the northern vorticity anomaly.

There are two critical aspects of this scenario. The first is that the difference in

the intrinsic phase speeds of the two anomalies is compensated by the different advections of the anomalies by the background flow, creating the possibility that the anomalies can be *phase locked* with one another. The second is that the anomalies can be *mutually amplifying*; i.e., each anomaly amplifies the other anomaly. These two aspects are critical to the process called *barotropic instability*. In the following section, we find analytic solutions to a particular example of a barotropically unstable flow.

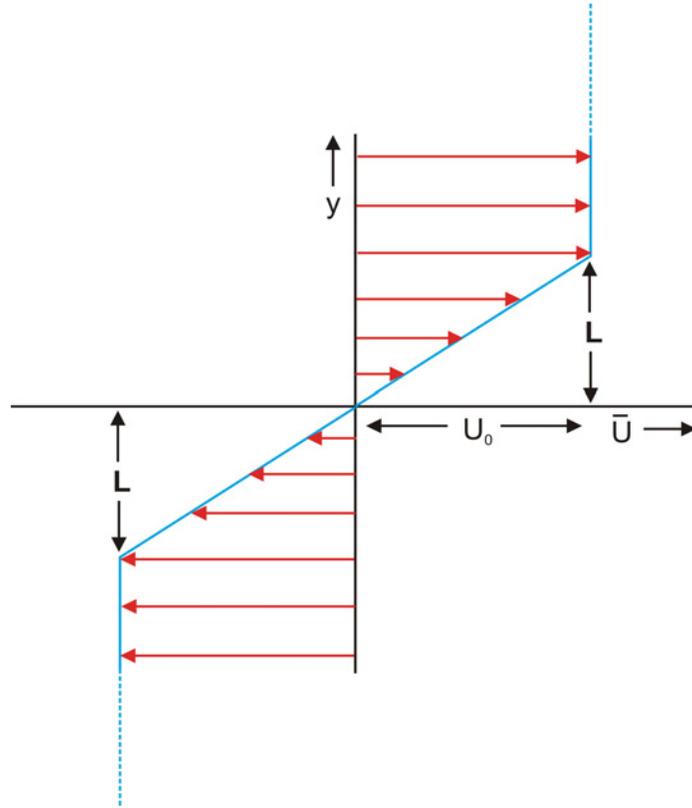


Figure 18.3

18.1 the Rayleigh problem

Consider the barotropic flow whose variation in y is illustrated in Figure 18.3.

The flow is piecewise continuous and all the meridional gradient of vorticity is concentrated in positive and negative delta functions at $y = +D$ and $y = -D$, respectively. We will formulate linear equations for small perturbations to this background state, solve them in each of the three regions, and match the solutions across the boundaries of the regions at $y = \pm D$.

The linearized momentum equations and mass continuity are as follows. Over-

bars signify the background state shown in Figure 18.3.

$$\frac{\partial u}{\partial t} + \bar{U} \frac{\partial u}{\partial x} + v \frac{\partial \bar{U}}{\partial y} = -\alpha_0 \frac{\partial p}{\partial x}, \quad (18.1)$$

$$\frac{\partial v}{\partial t} + \bar{U} \frac{\partial v}{\partial x} = -\alpha_0 \frac{\partial p}{\partial y}, \quad (18.2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (18.3)$$

Consider solutions of the form

$$u = \tilde{u}(y)e^{ik(x-ct)},$$

$$v = \tilde{v}(y)e^{ik(x-ct)},$$

$$p = \tilde{p}(y)e^{ik(x-ct)},$$

where c may be complex. Substitution into (18.1)–(18.3) gives

$$ik(\bar{U} - c)\tilde{u} + \bar{U}_y \tilde{v} = -\alpha_0 ik \tilde{p}, \quad (18.4),$$

$$ik(\bar{U} - c)\tilde{v} = -\alpha_0 \frac{d\tilde{p}}{dy}, \quad (18.5)$$

$$ik\tilde{u} + \frac{d\tilde{v}}{dy} = 0. \quad (18.6)$$

By cross-differentiating, we can eliminate \tilde{u} and \tilde{p} from (18.4)–(18.6) to arrive at a single O.D.E. in \tilde{v} :

$$\frac{d^2 \tilde{v}}{dy^2} - \tilde{v} \left(k^2 + \frac{\bar{U}_{yy}}{\bar{U} - c} \right) = 0. \quad (18.7)$$

Now note that in the interiors of each of the three regions in Figure 18.1, $\bar{U}_{yy} = 0$, so (18.7) reduces to

$$\frac{d^2 \tilde{v}}{dy^2} - k^2 \tilde{v} = 0 \quad (18.8)$$

within each region. Now we consider boundary conditions for solving (18.8). First we impose the condition that the solutions remain bounded at $y = \pm\infty$:

$$\lim_{y \rightarrow \pm\infty} \tilde{v} = \text{finite.}$$

General solutions of (18.8) that satisfy this condition are:

$$\begin{aligned} \text{I:} \quad \tilde{v} &= Ae^{-ky} \\ \text{II:} \quad \tilde{v} &= Be^{-ky} + Ce^{ky} \\ \text{III:} \quad \tilde{v} &= Fe^{ky} \end{aligned} \tag{18.9}$$

Next we apply boundary conditions at the boundaries separating the regions. There are two fundamental requirements:

- a. Fluid displacements must be continuous, and
- b. Pressure must be continuous.

The displacement in y , δy , is related to v by

$$v \equiv \frac{d}{dt} \delta y.$$

Linearizing this about the background state gives

$$v = \left(\frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) \delta y,$$

and substituting

$$\delta y = \tilde{\delta} y e^{ik(x-ct)},$$

we have

$$\tilde{v} = ik(\bar{U} - c)\tilde{\delta} y.$$

Continuity of $\tilde{\delta}y$ demands that

$$\frac{\tilde{v}}{ik(\bar{U} - c)} \text{ is continuous.} \quad (18.10)$$

In the present case, \bar{U} itself is continuous, so (18.10) implies that \tilde{v} is continuous.

Continuity of pressure implies, through (18.4), that the quantity

$$ik(\bar{U} - c)\tilde{u} + \bar{U}_y\tilde{v}$$

is continuous, or using (18.6) to eliminate $ik\tilde{u}$,

$$(\bar{U} - c)\frac{d\tilde{v}}{dy} - \bar{U}_y\tilde{v} \quad \text{is continuous.} \quad (18.11)$$

Matching \tilde{v} and the quantity given by (18.11) across each of the two boundaries in the general solutions (18.9) gives a condition on the relation between the complex phase speed c and k :

$$\left(\frac{Dkc}{U_0}\right)^2 = (Dk - 1)^2 - e^{-2kD}. \quad (18.12)$$

Here we see that c is purely real if $(Dk - 1)^2 \geq e^{-2kD}$, and purely imaginary if $(Dk - 1)^2 < e^{-2kD}$. In the former case, examination of (18.12) shows that to order $1/k$,

$$\lim_{k \rightarrow \infty} c = \pm U_0 \left[1 - \frac{2}{kD} \right]. \quad (18.13)$$

Very small-scale perturbations are confined to the delta function vorticity gradients at $y = \pm \frac{1}{2}D$ and move with the mean flow speed at the respective boundaries between regions, slightly slowed down owing the Rossby propagation effect.

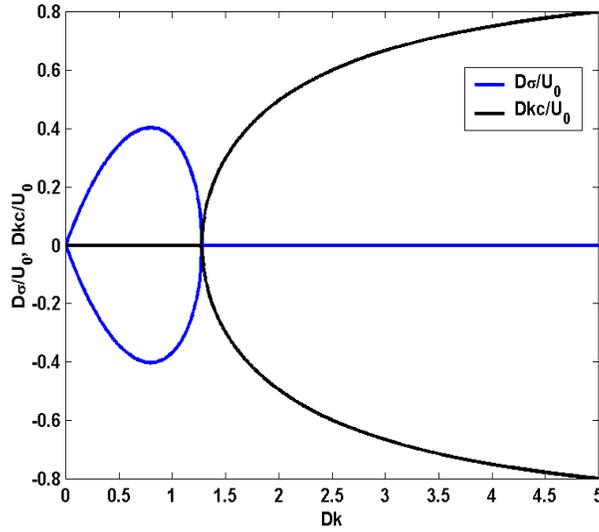


Figure 18.4

In the opposite limit of small k , we have to expand (18.12) to order k^2 to get

$$\lim_{k \rightarrow 0} c = \pm iU_0.$$

This denotes perturbations that do not propagate but which grow or decay at an exponential rate given by

$$\sigma \equiv kc_i,$$

so that

$$\lim_{k \rightarrow 0} \sigma = \pm kU_0. \quad (18.14)$$

For small k , the growth rate increases linearly with k .

Exact solutions of (18.12) are displayed in Figure 18.4.

Note the following:

1. $c = \sigma = 0$ when $Dk = 1.278$, corresponding to a wavelength $L \equiv \frac{2\pi}{k} = 4.92D$.

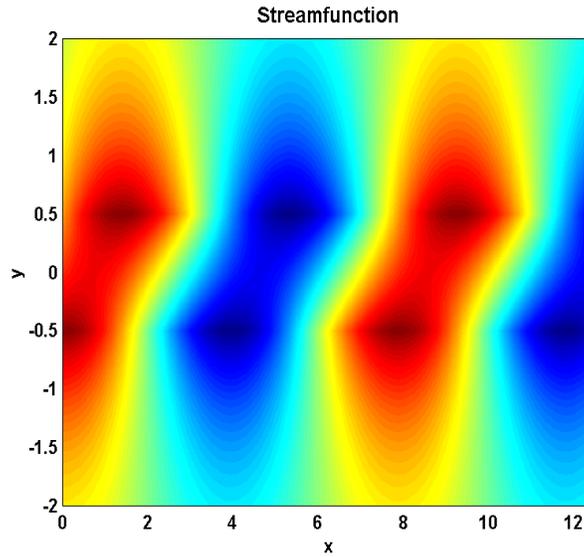


Figure 18.5a

2. The maximum (minimum) value of σ is $\pm 0.402U_0/D$ occurring when $Dk = 0.797$ ($L = 7.88D$).

Thus modal solutions fall into two classes: long waves that are stationary (and phase locked) and are either amplifying or decaying, and short neutral Rossby waves swimming upstream. These waves are too short to “feel” each other enough to become phase locked or mutually amplifying (or decaying).

The eigenmodes of the velocity streamfunction are shown in Figure 18.5 for $Dk = 0.797$, corresponding to the most rapidly growing mode, and for $Dk = 2.0$, corresponding to a stable mode on the southern vorticity gradient delta function. Note the “upshear” tilt of the unstable mode; eigenfunctions of the decaying mode tilt downshear.

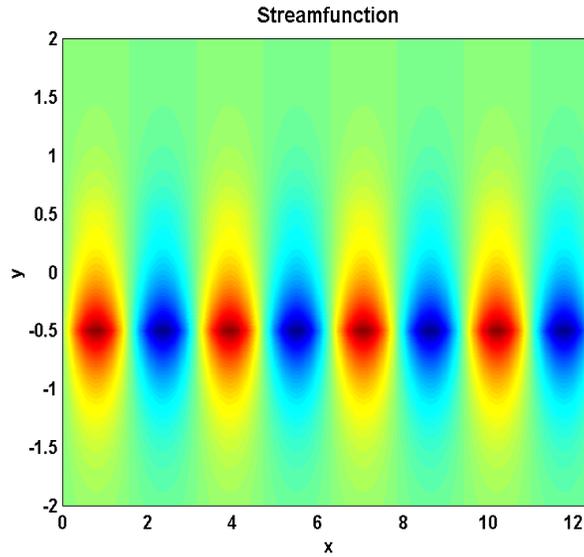


Figure 18.5b

18.2 Necessary conditions for barotropic instability

Rayleigh developed some general necessary conditions for instability of barotropic flows. These can also be stated as sufficient conditions for stability.

First suppose we have a barotropic flow $\bar{U}(y)$ in which the vorticity gradient is confined to some finite region, so that

$$\lim_{y \rightarrow \pm\infty} \frac{d^2\bar{U}}{dy^2} = 0.$$

For such a flow, we expect perturbations to vanish at $y = \pm\infty$ since the refractive index for wave propagation, the vorticity gradient, vanishes there.

Now consider modal disturbances to such a flow. These are governed by (18.7), for the meridional structure of the meridional wind. Now multiply (18.7) through by

the complex conjugate of \tilde{v} , \tilde{v}^* , and integrate the result through the whole domain:

$$\int_{-\infty}^{\infty} \left[\tilde{v}^* \frac{d^2 \tilde{v}}{dy^2} - |\tilde{v}|^2 \left(k^2 + \frac{\overline{U}_{yy}}{\overline{U} - c} \right) \right] dy = 0 \quad (18.15)$$

Here we have made use of the fact that

$$\tilde{v} \tilde{v}^* = |\tilde{v}|^2,$$

where $|\tilde{v}|$ is the absolute value of \tilde{v} . Now the first term in the integrand can be integrated by parts:

$$\int_{-\infty}^{\infty} \tilde{v}^* \frac{d^2 \tilde{v}}{dy^2} dy = \int_{-\infty}^{\infty} \frac{d}{dy} \left[\tilde{v}^* \frac{d\tilde{v}}{dy} \right] dy - \int_{-\infty}^{\infty} \left| \frac{d\tilde{v}}{dy} \right|^2 dy.$$

The first term on the right can be integrated exactly, but it vanishes because $\tilde{v} \rightarrow 0$ as $y \rightarrow \pm\infty$. Thus

$$\int_{-\infty}^{\infty} \tilde{v} \frac{d^2 \tilde{v}}{dy^2} dy = - \int_{-\infty}^{\infty} \left| \frac{d\tilde{v}}{dy} \right|^2 dy. \quad (18.16)$$

Using (18.16), we may write (18.15) as

$$\int_{-\infty}^{\infty} \left[\left| \frac{d\tilde{v}}{dy} \right|^2 + |\tilde{v}|^2 \left(k^2 + \frac{\overline{U}_{yy}}{\overline{U} - c} \right) \right] dy = 0. \quad (18.17)$$

Remember that c is, in general, complex, so the real and imaginary parts of (18.17) must both be satisfied. The imaginary part of (18.17) is

$$c_i \int_{-\infty}^{\infty} \frac{\overline{U}_{yy}}{|\overline{U} - c|^2} |\tilde{v}|^2 dy = 0, \quad (18.18)$$

where c_i is the imaginary part of c , which is positive for growing disturbances.

The relation (18.18) shows that one of two things must be true: Either

- a. $c_i = 0$, or
- b. the integral in (18.18) vanishes.

Thus we may conclude the following:

1. A *necessary* condition for instability ($c_i > 0$) is that \overline{U}_{yy} change sign at least once within the domain. In other words, the mean state vorticity must have an extremum in the domain. But note that even if \overline{U}_{yy} does change sign, this is no *guarantee* that the integral vanishes or that $c_i > 0$. This condition is *not* sufficient for instability.
2. If there is no extremum of vorticity within the domain, $c_i = 0$ and this is therefore a *sufficient* condition for stability.

Points 1 and 2 are really saying the same thing.

Another theorem, due to FjØtoft, may be derived by looking at the real part of (18.17):

$$\int_{-\infty}^{\infty} \frac{\overline{U}_{yy}(\overline{U} - c_r)}{|\overline{U} - c|^2} |\tilde{v}|^2 dy = - \int_{-\infty}^{\infty} \left[\left| \frac{d\tilde{v}}{dy} \right|^2 + k^2 |\tilde{v}|^2 \right] dy, \quad (18.19)$$

where c_r is the real part of c . Note that for *growing* disturbances, we are free to add any multiple of the integral in (18.18) to the left side of (18.19), since the former vanishes. We choose the multiplying factor to be c_r , giving

$$\int_{-\infty}^{\infty} \frac{\overline{U}\overline{U}_{yy}}{|\overline{U} - c|^2} |\tilde{v}|^2 dy = - \int_{-\infty}^{\infty} \left[\left| \frac{d\tilde{v}}{dy} \right|^2 + k^2 |\tilde{v}|^2 \right] dy. \quad (18.20)$$

Since the right-hand side of (18.20) is negative definite, so must the left side. So fluctuations of \overline{U} *must be negatively correlated with* \overline{U}_{yy} for growing disturbances.

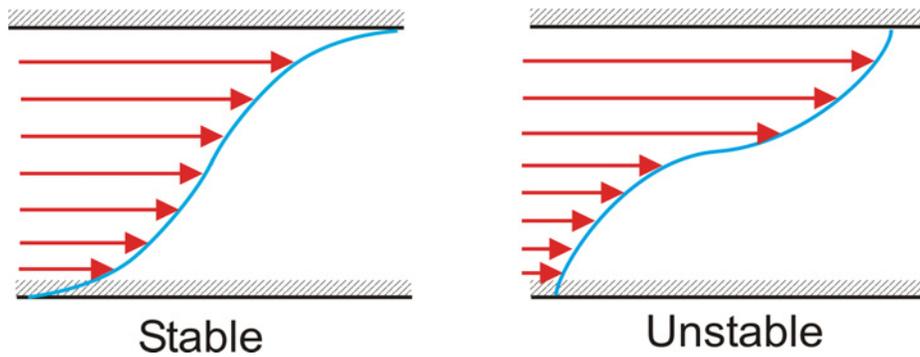


Figure 18.6

(Again, this is a necessary but not sufficient condition for instability.) Figure 18.6 shows an example of a flow that the Fjørtoft theorem shows to be stable in spite of satisfying the Rayleigh necessary condition for instability, and a flow which satisfies both necessary conditions for instability.

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