

Course 12.812, General Circulation of the Earth's Atmosphere  
 Prof. Peter Stone  
**Section 7: Forcing of Zonal Mean Circulations**

**Forcing of [u], [θ]:**

For simplicity and in order to include arbitrary eddy amplitudes, we will develop the equations using the quasi-geostrophic approximation and (fx, y, pressure) coordinates. However, the theoretical development can be generalized to the primitive equations for small amplitude eddies and to spherical coordinates (see Andrews & McIntyre, 1976; and Edmon et al, 1980<sup>1</sup>). The generalization to the primitive equations is not very useful for diagnostic calculations because it requires the introduction of vertical eddy fluxes which are difficult to measure, and assumes small eddy amplitudes.

What we are after is equations for the zonal mean circulations. The zonal mean of the

continuity equations is  $\frac{\partial [v]}{\partial y} + \frac{\partial [\omega]}{\partial p} = 0$ . Since  $[\omega] = O(Ro)$ ,  $[v] = O(Ro)$  too.  $\therefore$  we

need to develop the quasi-geostrophic equations with  $[u] = O(1)$ , and

$[v], [\omega], [\theta] = O(Ro)$ . One equation is the above continuity equation. Another is the geostrophic relation for u, which we will use in its thermal wind form, so as not to introduce the geopotential as another unknown. Since this is a linear relation, its zonal mean is unchanged: (R = gas constant)

$$f \frac{\partial [u]}{\partial p} = \frac{R}{p} \frac{\partial [\theta]}{\partial y}, \quad \theta = T - T_s(p), \quad T = \text{temperature}, \quad T_s = \text{basic state stratification.}$$

The quasi-geostrophic equation for  $\theta$  is

$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial x}(u\theta) + \frac{\partial}{\partial y}(v\theta) + \sigma\omega = \frac{Q}{C_p}$$

where u and v are geostrophic, and  $O(1)$ , and  $\sigma = \frac{dT_s}{dp} - \frac{RT_s}{C_p p}$ ,  $T_s = f(p)$  only

(N.B.,  $\sigma < 0$ ), and  $\omega$  and  $\theta$  are  $O(Ro)$ .  $\therefore v = [v]^0 + v^*$ ,  $\theta = [\theta] + \theta^*$ ,

and if we average zonally we obtain

$$\frac{\partial [\theta]}{\partial t} + \frac{\partial}{\partial y}[v^*[\theta] + v^*\theta^*] + [\omega]\sigma = \frac{[Q]}{C_p}.$$

---

<sup>1</sup> The assumption of small Rossby number also requires the assumption that

$\frac{\beta}{f} \lesssim \text{Rossby number.}$

Since  $[v^*] = 0$ , we can rewrite this as

$$\boxed{\frac{\partial [\theta]}{\partial t} + [\omega] \sigma = \frac{[Q]}{C_p} - \frac{\partial}{\partial y} [v^* \theta^*].}$$

For our fourth equation, we consider the quasi-geostrophic u equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial y} (uv) = fv - g \frac{\partial Z}{\partial x} + F.$$

Again u, v on the left are geostrophic and  $O(1)$  but on the right v must be accurate to  $O(Ro)$ . Now we average zonally and obtain

$$\boxed{\frac{\partial [u]}{\partial t} - f_0 [v] = [F] - \frac{\partial}{\partial y} [v^* u^*].}$$

(Note that since  $[v] = O(Ro)$ , to be consistent  $f[v] = f_0[v]$ .)

Now we have 4 equations for  $[u], [v], [\omega], [\theta]$ , and we see that this zonal mean circulation can be forced by diabaticity ( $[Q]$ ), dissipative stress ( $[F]$ ) or by eddy fluxes.

In fact, we can now state the Eliassen-Palm Theorem: In a steady, adiabatic, non-dissipative system,

$$\boxed{\frac{\partial}{\partial y} [u^* v^*] - \frac{\partial}{\partial p} \frac{f_0 [v^* \theta^*]}{\sigma} = 0.}$$

(Note that, if  $\Theta =$  potential temperature, then  $\frac{[v^* \theta^*]}{\sigma} = \frac{[v^* \Theta^*]}{\partial \Theta / \partial p}$ .)

To prove this we apply the stated conditions to the  $[u]$  and  $[\theta]$  equations, to obtain

$$[\omega] = \frac{\frac{\partial}{\partial y} [v^* \theta^*]}{\sigma} \quad \text{and} \quad [v] = \frac{1}{f_0} \frac{\partial}{\partial y} [u^* v^*].$$

(i.e. the effect of the eddies is just balanced by the effect of the MMC, so there is no net transport of heat or momentum.) Substituting these results into the continuity equation we then obtain

$$\frac{\partial}{\partial y} \frac{1}{f_0} \frac{\partial}{\partial y} [u^* v^*] - \frac{\partial}{\partial p} \frac{\frac{\partial}{\partial y} [v^* \theta^*]}{\sigma} = 0.$$

We integrate this with respect to  $y$ , and apply the B.C. that  $v = [v] = v^* = 0$  at the pole.

From the expression for  $[v]$  we see that this implies  $\frac{\partial}{\partial y}[u^* v^*] = 0$  at the pole. We then find that the integration function  $\equiv 0$  and obtain the Eliassen-Palm Theorem, Q.E.D.

This theorem motivates in part the definition of the Eliassen-Palm flux, a 2D flux in the meridional plane:

$$\vec{F} = \left( -[u^* v^*], \frac{f_0 [v^* \theta^*]}{\sigma} \right)$$

Thus  $\frac{\partial}{\partial t} [Q] = [F] = 0 \Rightarrow \nabla \cdot \vec{F} = 0$ ; another form of the theorem is:

$$[\bar{Q}] = [\bar{F}] = 0 \Rightarrow \overline{\nabla \cdot \vec{F}} = 0.$$

In the more general case of steady motion (or a stationary mean motion) with non-zero diabaticity and dissipation, we have

$$[\omega] = \frac{[Q]}{C_p \sigma} - \frac{\partial}{\partial y} \frac{[v^* \theta^*]}{\sigma}$$

$$[v] = -\frac{1}{f_0} [F] + \frac{1}{f_0} \frac{\partial [u^* v^*]}{\partial y},$$

$$\therefore \frac{\partial [v]}{\partial y} + \frac{\partial [\omega]}{\partial p} = -\frac{\partial [F]}{\partial y f_0} + \frac{\partial}{\partial y} \frac{1}{f_0} \frac{\partial [u^* v^*]}{\partial y} + \frac{\partial [Q]}{\partial p C_p \sigma} - \frac{\partial^2 [v^* \theta^*]}{\partial p \partial y \sigma} = 0$$

$$\therefore \frac{\partial [Q]}{\partial p C_p \sigma} - \frac{\partial}{\partial y} \frac{1}{f_0} ([F] + \nabla \cdot \vec{F}) = 0$$

i.e., now the convergence of the Eliassen-Palm flux must balance the diabaticity and dissipation.

Transformed Eulerian Mean Equations:

The above suggests that  $\vec{F}$  (or  $\nabla \cdot \vec{F}$ ) is a particularly convenient, and comprehensive measure of eddy forcing of the zonal mean state. In fact it is possible to transform the equations so that this is explicit. We define a residual mean circulation:

$$\tilde{\omega} = [\omega] + \frac{\partial}{\partial y} \frac{[v^* \theta^*]}{\sigma} \quad (\text{residual mean vertical circulation})$$

$$\tilde{v} = [v] - \frac{\partial [v^* \theta^*]}{\partial p \sigma} \quad (\text{residual mean meridional circulation})$$

(The basic definition of the residual mean circulation is that it is the circulation that represents the total advection of heat, or, it is the zonal mean circulation excluding the zonal mean circulation induced by the eddy heat fluxes:  $\tilde{\omega} = [\omega] - \omega_{\text{induced}}$ .)

This circulation preserves continuity:

$$\frac{\partial [v]}{\partial y} + \frac{\partial [\omega]}{\partial p} = \frac{\partial \tilde{v}}{\partial y} - \frac{\partial^2 [v^* \theta^*]}{\partial y \partial p \sigma} + \frac{\partial \tilde{\omega}}{\partial p} + \frac{\partial^2 [v^* \theta^*]}{\partial y \partial p \sigma} = 0$$

$$\therefore \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{\omega}}{\partial p} = 0.$$

This residual circulation has a simple interpretation for a steady state. In this case the  $\theta$  equation becomes:

$$[\omega] = \frac{[Q]}{C_p \sigma} - \frac{\partial [v^* \theta^*]}{\partial y \sigma} \Rightarrow \tilde{\omega} = \frac{[Q]}{C_p \sigma};$$

i.e.,  $\tilde{\omega}, \tilde{v}$  describe that part of the steady meridional circulation that arises from the zonal mean diabatic heating. Therefore the  $\theta$  equation is simply:

$$\frac{\partial [\theta]}{\partial t} + \sigma \left( \tilde{\omega} - \frac{\partial [v^* \theta^*]}{\partial y \sigma} \right) = \frac{[Q]}{C_p} - \frac{\partial [v^* \theta^*]}{\partial y}, \text{ or}$$

$$\frac{\partial [\theta]}{\partial t} + \tilde{\omega} \sigma = \frac{[Q]}{C_p}$$

The thermal wind equation is unaffected by our change,  $f \frac{\partial [u]}{\partial p} = \frac{R}{p} \frac{\partial [\theta]}{\partial y}$ , and the u equation becomes

$$\frac{\partial [u]}{\partial t} - f_0 \left( \tilde{v} + \frac{\partial [v^* \theta^*]}{\partial p \sigma} \right) = [F] - \frac{\partial [v^* u^*]}{\partial y}, \text{ or}$$

$$\frac{\partial [u]}{\partial t} - f_0 \tilde{v} = [F] + \nabla \cdot \vec{F}$$

The transformed equations show explicitly that  $[u]$  and  $[\theta]$  (and also  $\tilde{v}, \tilde{\omega}$ ) are forced by the eddies only to the extent that  $\nabla \cdot \vec{F} \neq 0$ .

This is a consequence of the non-acceleration theorem of Charney and Drazin (1961):

$$\text{If } [F] = [Q] = \nabla \cdot \vec{F} = 0 \text{ then there exists a trivial solution } \frac{\partial [u]}{\partial t} = \frac{\partial [\theta]}{\partial t} = \tilde{v} = \tilde{\omega} = 0.$$

Thus  $\nabla \cdot \vec{F}$  represents the total explicit internal eddy forcing of the  $[u]$  and  $[\theta]$  fields.

There could also be external forcing associated with the boundary conditions, or implicit forcing associated with eddy modifications of  $[F]$  and  $[Q]$ .

Also we note that for time averages, for quasi-geostrophic motion with  $[v] = 0$ ,

$$[\overline{uv}] = [\overline{u * v *}] = [\overline{u} * \overline{v} *] + [\overline{u'v'}], \text{ etc.,}$$

i.e. the eddy forcing can be broken into stationary and transient components.

#### Quasi Geostrophic Potential Vorticity:

Our quasi geostrophic  $\beta$  plane equations are (in  $x, y, p$  coordinates)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = f v - g \frac{\partial Z}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -f u - g \frac{\partial Z}{\partial y} + F_y$$

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + \sigma \omega = \frac{Q}{C_p};$$

Suppose we have frictionless, adiabatic flow. If we cross differentiate the first two equations, simplify and invoke geostrophy for the advection terms, i.e.,  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ ,

we get (remembering we must keep  $O(\text{Ro})$  contributions from the Coriolis terms)

$$\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} = 0 \text{ where}$$

$$q = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f + f_0 \frac{\partial \theta}{\partial p \sigma} = \text{quasi-geostrophic potential vorticity (PV)}.$$

Now consider the zonal mean of the PV equation, again invoking continuity and quasi geostrophy:

$$\therefore \frac{\partial [q]}{\partial t} + \frac{\partial [vq]}{\partial y} = 0; \text{ but } [vq] = [v * q *]$$

$$\therefore \frac{\partial [q]}{\partial t} = -\frac{\partial [v * q *]}{\partial y}; \text{ i.e. } [q] \text{ is simply forced by } [v * q *].$$

However, we can relate  $[v^*q^*]$  to the other eddy fluxes. Substituting from the definition of  $q$ , we have

$$[v^*q^*] = \left[ v^* \frac{\partial y^*}{\partial x} \right] - \left[ v^* \frac{\partial u^*}{\partial y} \right] + f \left[ v^* \right] + f_0 \left[ v^* \frac{\partial \theta^*}{\partial p \sigma} \right]; \text{ but}$$

$$\left[ v^* \frac{\partial u^*}{\partial y} \right] = \frac{\partial}{\partial y} [u^* v^*] - \left[ u^* \frac{\partial v^*}{\partial y} \right] \text{ since } \frac{\partial v^*}{\partial y} = -\frac{\partial u^*}{\partial x} \text{ and}$$

$$\left[ v^* \frac{\partial \theta^*}{\partial p \sigma} \right] = \frac{\partial}{\partial p} \left[ \frac{v^* \theta^*}{\sigma} \right] - \left[ \frac{\theta^* \partial v^*}{\sigma \partial p} \right]$$

$$\text{since } f \frac{\partial v^*}{\partial p} = -\frac{R}{p} \frac{\partial \theta^*}{\partial x} \text{ (thermal wind relation)}$$

$$\therefore [v^*q^*] = -\frac{\partial}{\partial y} [u^* v^*] + f_0 \frac{\partial}{\partial p} \left[ \frac{v^* \theta^*}{\sigma} \right]$$

$$\therefore \boxed{[v^*q^*] = \nabla \cdot \vec{F}}$$

Thus the total explicit eddy forcing is given simply by the eddy PV flux.

**Observations:** Edmon et al (1980) calculated  $\vec{F}$  and  $\nabla \cdot \vec{F}$  from the analysis of Oort & Rasmussen (1971) for winter and summer, and the results are shown in Figure 5 in Edmon et al (1980). We see that the eddy forcing in winter is characterized by a strong divergence in very low levels in mid-latitudes and convergence in the middle and upper troposphere in mid-latitudes. In the earlier view, in which  $[\bar{u}]$  is forced only by

$[\overline{u^* v^*}]$ , only the upper level divergence is taken into account. In the more modern

view, based on  $\nabla \cdot \vec{F}$ ,  $[v^* \theta^*]$  dominates. This tends to decrease  $\frac{\partial \bar{u}}{\partial z}$  and  $\int \bar{u}^2 dp$ ; i.e.,

the total effect of the eddies is to decrease the mean kinetic energy. The forcing is much weaker in summer (note the different contour interval), and the main region of convergence has shifted down to the ground. Note that the analyses are from 1000mb up. Since  $[v^* \theta^*]$  is still strong at 1000mb, but must be zero at the surface, there must be a very strong  $\nabla \cdot \vec{F} > 0$  between the surface and 1000mb which is not captured by the analysis.

**Generalization to include eddy forcing of condensation** (see Stone and Salustri, 1984)

The above discussion does not take into account that the eddies force the moisture field, i.e., there is an eddy latent heat flux. The definitions and equations can be generalized to take this into account, as long as one is willing to make the quasi-geostrophic approximation. We write the moisture conservation equation in pressure coordinates:

$$\frac{\partial q}{\partial t} + \nabla \cdot (\bar{v}q) = -C$$

Now let  $\bar{v} = \bar{v}_g + \bar{v}_a$ , where  $\bar{v}_g$  represents the large scale component of the motions, which can be approximated by the quasi-geostrophic approximation.  $\bar{v}_a$  is the remainder velocity field, i.e., the smaller scale velocity field. Let  $S = -\nabla \cdot \bar{v}_a q$  = convergence of moisture associated with the smaller scale motions.  $\therefore S$  = an apparent moisture source for the large scale motions. Now we simplify  $\nabla \cdot \bar{v}_g q$  by using the quasi-geostrophic approximation to obtain an equation analogous to the quasi-geostrophic thermodynamic equation. If we average zonally and drop the subscript g, let

$q = q_s(p) + q'$ , where  $q_s = \text{mean } q = O(1)$ ,  $q' = \text{remainder} = O(\text{Ro})$ ,  $q'^* = q^*$ , then

$$\begin{aligned} \frac{\partial}{\partial y} [vq] + \frac{\partial}{\partial p} [\omega q] &= \frac{\partial}{\partial y} ([v]q_s + [v^* q'^*]) + \frac{\partial}{\partial p} ([\omega]q_s) + O(\text{Ro}^2) \\ &= q_s(p) \left( \frac{\partial [v]}{\partial y} + \frac{\partial [\omega]}{\partial p} \right) + \frac{\partial}{\partial y} [v^* q'^*] + [\omega] \frac{\partial q_s}{\partial p}, \text{ and} \end{aligned}$$

$$\therefore \frac{\partial [q']}{\partial t} + \frac{\partial}{\partial y} [v^* q'^*] + [\omega] \frac{\partial q_s}{\partial p} = -[C - S].$$

(Thus we have assumed  $q' = O(\text{Ro})$  and neglected terms involving  $[v]$  and

$[\omega^* q'^*]$  because they are  $O(\text{Ro}^2)$ , but retained  $\frac{\partial q_s}{\partial p} [\omega]$  because  $\frac{\partial q_s}{\partial p} \sim 1$ , just as in the quasi-geostrophic thermodynamic equations. This is not as good an approximation as for  $T_s$ , because  $\frac{\partial q_s}{\partial y}$  is relatively stronger.)

We see that in a stationary state  $C-S$  is the condensation rate attributable to just the larger scale (quasi-geostrophic) motions.

Next we define  $Q_m = Q - L_v(C - S)$ ,  $\therefore$  in a stationary state  $Q_m$  is the net diabatic heating excluding the condensation heating attributable to the larger scale motions. Substituting into the thermodynamic equation for  $Q$  and  $C - S$ , we obtain

$$\frac{\partial}{\partial t} \left[ \theta + \frac{L_v}{C_p} q' \right] + \sigma_m [\omega] = \frac{[Q_m]}{C_p} - \frac{\partial}{\partial y} \left[ v^* \theta^* + \frac{L_v}{C_p} v^* q'^* \right]$$

where  $\sigma_m = \sigma + \frac{L}{C_p} \frac{\partial q_s}{\partial p}$  = “moist” static stability.

Now we define generalized residual mean circulations (and neglect y variations in  $\sigma_m$ ):

$$\tilde{v}_m = [v] - \frac{\partial}{\partial p} \frac{\left[ v^* q^* + \frac{L_v}{C_p} v^* q^* \right]}{\sigma_m} \equiv [v] - v_f$$

$$\tilde{\omega}_m = [\omega] + \frac{\partial}{\partial y} \frac{\left[ v^* \theta^* + \frac{L_v}{C_p} v^* q^* \right]}{\sigma_m} \equiv [\omega] - \omega_f$$

and a generalized Eliassen-Palm flux:

$$\bar{F}_m = (F_\phi^m, F_p^m), \quad F_\phi^m = -[u^* v^*]$$

$$F_p^m = \frac{f_0}{\sigma_m} \left[ v^* \theta^* + \frac{L_v}{C_p} v^* q^* \right]$$

(N.B. both  $\sigma \rightarrow \sigma_m$  and the addition of  $v^* q^*$  tend to increase  $F_p$ ;  $\sigma < 0$  for static stability.)  $v_f$  and  $\omega_f$  are mean circulations induced by the eddy heat fluxes.

Now we substitute into our equations and obtain an unchanged thermal wind relation, a continuity equation for  $\tilde{v}_m, \tilde{\omega}_m$ , plus:

$$\frac{\partial [u]}{\partial t} = f_0 \tilde{v}_m + [F] + \nabla \cdot \bar{F}_m$$

$$\frac{\partial}{\partial t} \left[ \theta + \frac{L_v}{C_p} q' \right] + \sigma_m \tilde{\omega}_m = \frac{[Q_m]}{C_p}$$

$$\frac{\partial [q']}{\partial t} + \frac{\partial q_s}{\partial p} \tilde{\omega}_m = [S - C] + M$$

where 
$$M = -\frac{\partial}{\partial y} [v^* q^*] + \frac{1}{\sigma_m} \frac{\partial q_s}{\partial p} \frac{\partial}{\partial y} \left[ v^* \theta^* + \frac{L_v}{C_p} v^* q^* \right]$$

$\therefore$  in effect we now have three evolution equations in place of two, but we have again reduced the number of explicit eddy forcing functions, this time from three ( $[v^* \theta^*], [u^* v^*], [v^* q^*]$ ) to two ( $\nabla \cdot \bar{F}_m$  and  $M$ ).

Note that we can rewrite the eddy forcing terms as:

$$\nabla \cdot \bar{F}_m = -\frac{\partial}{\partial y} [u^* v^*] + f_0 v_f \quad \text{and}$$

$$M = -\frac{\partial}{\partial y} [v^* q^*] - \omega_f \frac{\partial q_s}{\partial p}$$

One can state parallel forms of the Eliassen-Palm and non-acceleration theorems, but it is clear that now  $\nabla \cdot \bar{F}_m$  represents the explicit, internal eddy forcing of  $[u]$  whereas  $M$  represents the explicit internal eddy forcing of  $[q']$ . Both indirectly force  $[\theta]$  since it is coupled to  $[u]$  and  $[q']$  through the other equations.

Calculations of  $\nabla \cdot \bar{F}_m$  from Oort and Rasmussen's (1971) analysis are shown in figures 1 through 4 in Stone and Salustri (1984) for January and July.  $\sigma_m$  was evaluated at 50N; results at  $\phi \leq 30N$  may be unreliable. Also note that  $[v^* q^*]$  is strong at 1000mb (it even peaks there in the subtropics), so again there is a strong implied divergence between 1000mb and the surface.  $\nabla \cdot \bar{F}$  for the same months is shown for comparison. In January, the moisture effect increases the convergence considerably, introducing a new maximum convergence near 30N, 1000mb, which is more than double the maximum convergence with moisture effects omitted.  $\therefore$  the effect of the eddies to decelerate the jet is greatly increased, especially below the jet. In July, both the convergence and divergence are greatly increased by including the moisture effects, by 3 times and 8 times respectively. In fact the divergence maximum is now stronger than in January.

Clearly the moisture effects are a very important part of the eddy forcing of  $[\bar{u}]$ . In fact, we see that the effect due to  $[u^* v^*]$  (the upper level divergence maximum in the diagrams) is almost completely lost in the overall forcing, which is dominated by the eddy heat fluxes.

Also shown in figures 11 and 12 in Stone and Salustri (1984) is the eddy forcing of the moisture field, the function  $M$ , again calculated from Oort & Rasmussen's 1971 analyses for January and July. We note that  $M$  contains two contributions. The first is the conventional effect of the convergence of the eddy moisture flux, while the second is the effect of the mean vertical motion,  $\omega_f$ , induced by the eddy heat fluxes, i.e., the Ferrel Cell induced by these fluxes. Since the Ferrel Cell subsidence (drying out) coincides with divergence of the eddy moisture flux (also drying out) while the rising branch coincides with convergence of  $[v^* q^*]$ , the forcing is much stronger than it would appear from just the effect of  $[v^* q^*]$ : 2 times stronger in January and 4 times stronger in July. In general, the eddies tend to dry out the subtropics and moisten mid and high latitudes, and the effect is concentrated in the lower troposphere.

### **Forcing of the mean meridional circulation:**

A diagnostic equation, i.e. one with no explicit time derivative, can be derived for the mean meridional circulations, because of the thermal wind relation. First we define stream functions:

$$\left[ \begin{array}{l} [v] = \frac{\partial \psi}{\partial p}, \quad [\omega] = -\frac{\partial \psi}{\partial y} \\ \tilde{v} = \frac{\partial \tilde{\psi}}{\partial p}, \quad \tilde{\omega} = -\frac{\partial \tilde{\psi}}{\partial y} \end{array} \right]$$

From the definition of  $\tilde{v}$  we have

$$\tilde{v} = \frac{\partial \tilde{\psi}}{\partial p} = [v] - \frac{\partial [v * \theta *]}{\partial p \sigma} = \frac{\partial \psi}{\partial p} - \frac{\partial [v * \theta *]}{\partial p \sigma}$$

$$\therefore \psi = \tilde{\psi} + \frac{[v * \theta *]}{\sigma}$$

Now we derive an equation for  $\tilde{\psi}$ : Take  $\frac{R}{p} \frac{\partial}{\partial y}$  of the  $\theta$  equation:

$$\frac{R}{p} \frac{\partial}{\partial t} \frac{\partial [\theta]}{\partial y} + \frac{R}{p} \frac{\partial}{\partial y} (\sigma \tilde{\omega}) = \frac{R}{C_p p} \frac{\partial [Q]}{\partial y};$$

Take  $f_0 \frac{\partial}{\partial p}$  of the  $u$  equation:

$$f_0 \frac{\partial}{\partial t} \frac{\partial [u]}{\partial p} - f_0^2 \frac{\partial \tilde{v}}{\partial p} = f_0 \frac{\partial [F]}{\partial p} + f_0 \nabla \cdot \frac{\partial \bar{F}}{\partial p};$$

Subtract; cancel out the first terms from each equation (thermal wind relation); substitute for  $\tilde{v}$ ,  $\tilde{\omega}$  in terms of  $\tilde{\psi}$ ; and note that  $\sigma = -|\sigma|$ .

We obtain

$$\left[ f_0^2 \frac{\partial^2 \tilde{\psi}}{\partial p^2} + \frac{R}{p} \frac{\partial}{\partial y} \left( |\sigma| \frac{\partial \tilde{\psi}}{\partial y} \right) = \frac{R}{C_p p} \frac{\partial [Q]}{\partial y} - f_0 \frac{\partial [F]}{\partial p} - f_0 \nabla \cdot \frac{\partial \bar{F}}{\partial p} \right]$$

Now we substitute for  $\tilde{\psi}$  to obtain a  $\psi$  equation. Its eddy forcing may be written as

$$\left( f_0^2 \frac{\partial^2}{\partial p^2} + \frac{R}{p} \frac{\partial}{\partial y} |\sigma| \frac{\partial}{\partial y} \right) \frac{[v * \theta *]}{\sigma} - f_0 \nabla \cdot \frac{\partial \bar{F}}{\partial p};$$

$$\text{but } f_0 \nabla \cdot \frac{\partial \bar{F}}{\partial p} = -f_0 \frac{\partial^2}{\partial y \partial p} [u * v *] + f_0 \frac{\partial^2}{\partial p^2} \frac{f_0 [v * q *]}{\sigma};$$

∴ the first and last terms cancel, and we have

$$f_0^2 \frac{\partial^2 \psi}{\partial p^2} + \frac{\partial}{\partial y} \left( |\sigma| \frac{\partial \psi}{\partial y} \right) = \frac{R}{C_p p} \frac{\partial [Q]}{\partial y} - f_0 \frac{\partial [F]}{\partial p} - \frac{R}{p} \frac{\partial}{\partial y} \left( |\sigma| \frac{\partial \left( \frac{[v^* q^*]}{|\sigma|} \right)}{\partial y} \right) + f_0 \frac{\partial^2 [u^* v^*]}{\partial y \partial p}.$$

Note that these equations hold at any instant in time. The equation for  $\psi$  is frequently written in terms of  $[\omega]$  and called the omega equation. Because the equations are linear, the forcing by the diabatic heating, friction, and eddies can be viewed as giving rise to separate circulations which can be linearly added together.

The part due to the eddy forcing can be analyzed qualitatively beforehand, for the mean state, by looking at the simplified (stationary) forms of the  $\theta$  and  $u$  equations. The first is

$$[\omega] = \frac{-[Q]}{C_p |\sigma|} + \frac{\partial [v^* \theta^*]}{\partial y |\sigma|}$$

Recall that  $\omega = \frac{dp}{dt}$ , i.e.  $\omega < 0 \Rightarrow$  upward motion, etc., and warming,  $[Q] > 0$ , gives rise to rising motion, and cooling gives rise to sinking motion. We see that the  $[v^* \theta^*]$  term

is just an apparent heat source, with  $-\frac{\partial [v^* \theta^*]}{\partial y}$  being analogous to  $Q$ . ∴ the observed divergence of the eddy heat flux in low latitudes gives rise to cooling and sinking motions, while the observed convergence in high latitudes gives rise to warming and rising motions. Therefore the eddy heat flux forces a Ferrel Cell.

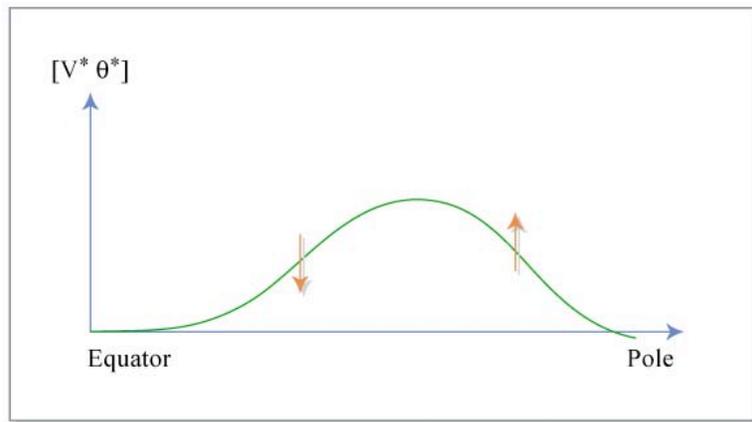


Figure by MIT OCW.

The stationary  $u$  equation is  $[v] = -\frac{1}{f_0} [F] + \frac{1}{f_0} \frac{\partial [u^* v^*]}{\partial y}$ ;

$\therefore -\frac{\partial}{\partial y}[u^* v^*]$  is analogous to  $[F]$ , the “frictional” stress on the zonal flow.

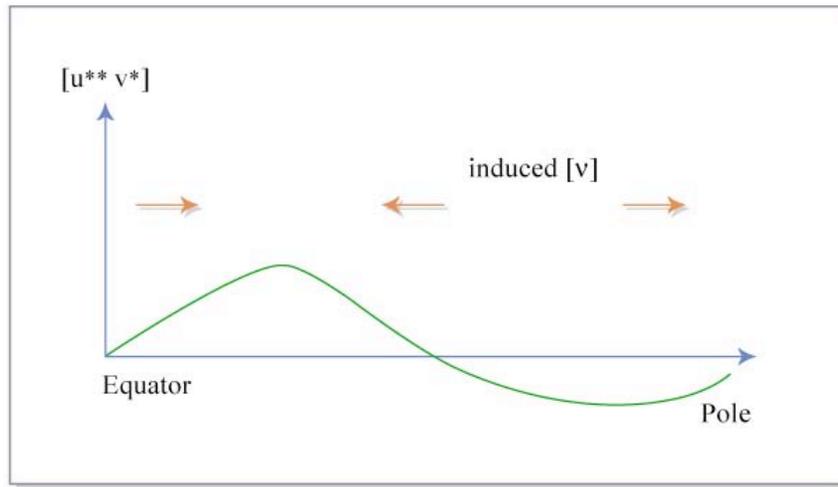


Figure by MIT OCW.

Thus the observed convergence of eddy momentum flux in middle latitudes at high levels is analogous to a positive  $F$ , i.e. an acceleration of the zonal flow, and the Coriolis force acting on this accelerated  $u$  will drive an equatorward component of the meridional velocity at high levels in mid-latitudes. Similarly the divergences in low and high latitudes will cause a (relatively weaker) poleward flow at high levels. Thus the eddy momentum flux tends to drive a three celled circulation dominated by a Ferrel Cell. Thus we see that the Ferrel Cell can be explained by the eddy forcing.

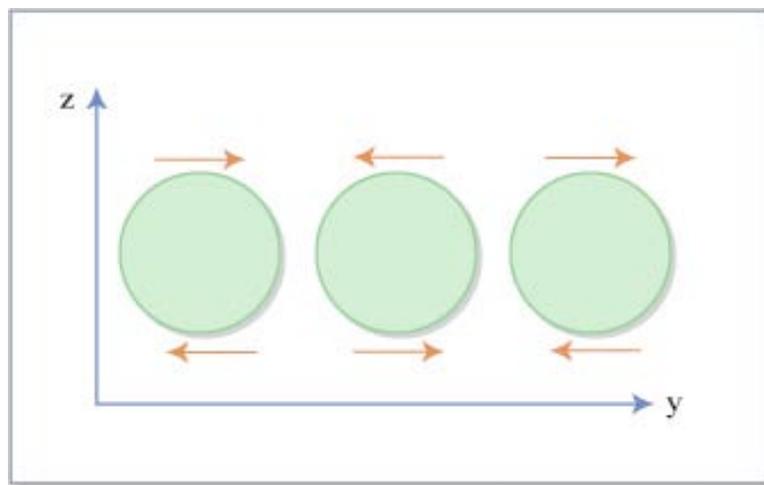


Figure by MIT OCW.

One calculation of the Ferrel Cell forced by these two eddy effects is shown in Figures 2, 3, 10 and 11 in Vernekar (1967). (These are from NMC data for a single January and July.) The Ferrel Cell is quite clear, and much stronger in January than July.

The component of the MMC due to diabatic heating, as we saw above, is just the residual MMC. Edmon et al, (1980) and Stone and Salustri (1984) also calculated the stream function for this circulation, from the definition of  $\tilde{v}$  :

$$\tilde{v} = [v] - \frac{\partial}{\partial p} \frac{[v * \theta *]}{\sigma} = \frac{\partial \tilde{\psi}}{\partial p}$$

by using Oort & Rasmussen's (1971) data for  $[v]$  (the balanced  $[v]$ ) and  $[v * \theta *]$  and integrating downward from  $p = 0$  with the B.C.  $\psi = 0$ . The results for January and July are shown in Figures 7 and 9 in Stone and Salustri (1984). As we would expect, the result is dominated by a strong Hadley Cell with a strong seasonal shift. We note that a residual Ferrel Cell is also apparent, either because of ageostrophic effects or because the eddy moisture flux forcing is still included in  $Q$ .

All the above can be again generalized to include the effect of eddies on condensation by replacing  $Q$  by  $Q_m$ . The effect on  $\psi$  is very straightforward:  $[v * q *]$  merely enhances  $[v * \theta *]$  i.e., it too tends to drive a Ferrel Cell by cooling low latitudes and warming high latitudes. However the effect is relatively weak: negligible in January, while in July  $[\omega]$  is enhanced by ~30% (Salustri and Stone, 1983, JAS, 40, 1101). This is because  $[u * v *]$  makes such a large contribution to the Ferrel Cell. However, the effect on  $\tilde{\psi}$  is much more substantial, since  $[u * v *]$  does not affect this. The result for  $\tilde{\psi}$  when eddy forcing of  $Q$  is taken into account is shown in Figures 8 and 10 in Stone and Salustri (1984) where it may be compared with the “dry” result. In both January and July the Hadley cell forced by  $Q$  in mid-latitudes is greatly enhanced by removing the eddy forcing of condensation from  $Q$ , and the residual Ferrel Cell is correspondingly reduced.

A diagnostic calculation of the MMC or  $[u]$  forced by friction is more difficult, since frictional effects cannot be directly measured, and we don't have any simple relation like that involving  $\tilde{\omega}$  and  $Q$ . (Of course, a residual calculation is possible, but then we are not really “explaining” the observed MMC.) Also, there is no alternate definition of residual MMC which gives us a simple expression for  $[v]$  forced by  $[F]$ . However, there are parameterizations of  $F$  available that could enable one to make reasonable estimates of  $F$  from other fields, such as the surface winds. In any case, we note that the main source of dissipation in the atmosphere is surface drag, which opposes the surface wind. From our above relations, we see that in balance,

$$[\bar{v}] = -\frac{1}{f}[\bar{F}] + \frac{\partial}{\partial y}[u * v *].$$

Therefore near the surface, in low latitudes where there are

easterlies,  $[F] > 0 \Rightarrow [v] < 0$ ; in mid latitudes where there are westerlies,  $[v] > 0$ , etc. ∴

we expect surface frictional effects to reinforce the mid-latitude Ferrel Cell and the direct cells in low and high latitudes (see figure on next page). Nobody has yet calculated in a

realistic way the size of this effect. (Crawford and Sasamori, 1981, *Tellus*, 33, 340 made a calculation but assumed that the decay time associated with surface drag is only one day, which is unrealistically large).

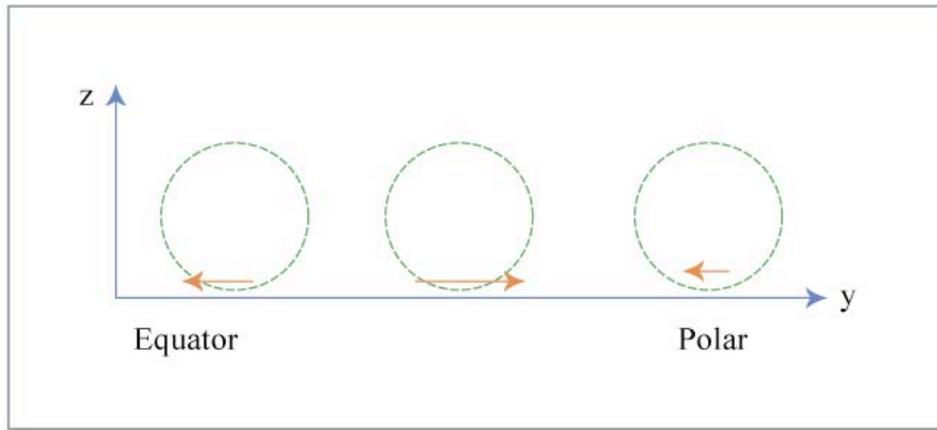


Figure by MIT OCW.