12.864 Inference from Data and Models   4 April 2003
Problem Set No. 3   Due: 18 April 2003

1. Using an eigenvector/eigenvalue analysis, solve (a)

$$\left\{\begin{array}{ccc} 1 & 1 & -2 \\ 1 & 2 & -1 \\ -2 & -1 & 6 \end{array}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tag{1}$$

and (b)

$$\left\{\begin{array}{ccc} 1 & 1 & -2 \\ 1 & 2 & -1 \\ 1.5 & 2 & -2.5 \end{array}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

2. (a) Find the ranges and null spaces of

$$\mathbf{A} = \left\{\begin{array}{ccc} 2 & -1 & 1 \\ 3 & 2 & 1 \end{array}\right\} \tag{3}$$

and calculate the solution and data resolution matrices. (b) Let there be a set of observations $\mathbf{y}$, such that

$$\mathbf{Ax} + \mathbf{n} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{4}$$

This problem is clearly formally undetermined. Find the solution which minimizes

$$J = \mathbf{x}^T\mathbf{x} \tag{5}$$

and compare it to the SVD solution with null space set to zero. What is the uncertainty of this solution? (You will need to assume something plausible about the noise variance.) (c) Now consider instead

$$\mathbf{A} = \left\{\begin{array}{cc} 2 & 3 \\ -1 & 2 \\ 1 & 1 \end{array}\right\}, \tag{6}$$

and the formally overdetermined problem

$$\mathbf{Ax} + \mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \tag{7}$$

and find the least-squares solution which minimizes $\mathbf{n}^T\mathbf{n}$. What is the uncertainty of this solution? How does the solution compare to the SVD solution? (d) For an arbitrary $\mathbf{A}$, solve the least-squares problem of minimizng

$$J = \mathbf{x}^T\mathbf{x} + \alpha^{-2}\mathbf{n}^T\mathbf{n} \tag{8}$$

and re-write the solution in terms of its SVD. Discuss what happens to the small singular value contributions.

3. There is one observation

$$x + n_1 = 1 \tag{9}$$

and a priori statistics $< n >=< x >= 0, < n^2 >= 1/2, < x^2 >= 1/2$. (a)What is the best estimate of x, n? (b) A second measurement becomes available,

$$x + n_2 = 3 \tag{10}$$

with $< n_2 >= 0, < n_2^2 >= 4$. What is the new best estimate of x and what is its estimated uncertainty. Are the various a priori statistics consistent with the final result?

4. Two observations of unknown x produce the apparent results

$$x = 1 \tag{11}$$

$$x = 3 \tag{12}$$

Produce a reasonable value for x under the assumption that (a) both observations are equally reliable, and (b) that the second observation is much more reliable (but not infinitely so) than the first (make some reasonable numerical assumption about what "reliable" means and state what you are doing). Can you re-write eqs. (11,12) in a more sensible form?

5. For the Laplace-Poisson equation $\nabla^2\phi = \rho$ with Dirichlet boundary conditions in a square domain, put it into discrete form and code it on a computer so that it can be written,

$$\mathbf{Ax} = \mathbf{b}. \tag{13}$$

Choose any reasonable dimension for the number of grid points or finite elements or basis functions. Confirm that $\mathbf{A}$ is square. (a) For any reasonable boundary conditions ϕ_b and values of ρ, solve (13) as a forward problem (b) Add some random noise to ϕ_b and solve it again. (c) Omit any knowledge of ρ over some part of the domain and find at least one possible solution (you could use least-squares). (d) Omit any knowledge of ϕ_b over some part of the domain and find at least one possible solution. (e) Suppose ϕ from (a) is known over part of the domain, use that knowledge to help improve the solutions in (b-d).

6. The temperature along an oceanic transect is believed to satisfy a linear rule, $\theta = ar + b$,where r is the distance from a reference point, and a, b are constants. Measurements of θ at sea, called y, produce the following values,$r = 0, y = 10; r = 1, y = 9.5; r = 2, y = 11.1, r = 3, y = 12$. (a) Using ordinary least-squares, find an estimate of a, b and the noise in each measurement, and their standard errors. (b) Solve it again using the SVD and discuss, via the resolution matrices, which of the observations, if any proved most important. Is the solution fully resolved?