

Contents

Preface	ix
I Fundamental Machinery	1
1 Introduction	1
1.1 Differential Equations	2
1.2 Partial Differential Equations	5
1.3 More Examples	8
1.4 Importance of the Forward Model	15
2 Basic Machinery	19
2.1 Background	19
2.2 Matrix and Vector Algebra	19
2.2.1 Matrix Multiplication and Identities	23
2.2.2 Linear Simultaneous Equations	24
2.2.3 Matrix Norms	26
2.2.4 Identities. Differentiation.	27
2.3 Simple Statistics. Regression	29
2.3.1 Probability Densities, Moments.	29
2.3.2 Sample Estimates. Bias.	31
2.3.3 Multivariable Probability Densities. Correlation	32
2.3.4 Functions and Sums of Random Variables	39
2.4 Least-Squares	42
2.4.1 Basic Formulation	43
2.4.2 Weighted and Tapered Least-Squares	51
2.4.3 Underdetermined Systems and Lagrange Multipliers	58
2.4.4 Interpretation of Discrete Adjoints	68

2.5	The Singular Vector Expansion	69
2.5.1	Simple Vector Expansions	70
2.5.2	Square-Symmetric Problem. Eigenvalues/Eigenvectors	72
2.5.3	Arbitrary Systems	84
2.5.4	The Singular Value Decomposition	90
2.5.5	Some Simple Examples. Algebraic Equations.	93
2.5.6	Simple Examples. Differential and Partial Differential Equations	101
2.5.7	Relation of Least-Squares to the SVD	105
2.5.8	Pseudo-Inverses	107
2.5.9	Row and Column Scaling	107
2.5.10	Solution and Observation Resolution. Data Ranking	113
2.5.11	Relation to Tapered and Weighted Least-Squares	115
2.5.12	Resolution and Variance of Tapered Solutions	118
2.6	Combined Least-Squares and Adjoints	119
2.6.1	Exact Constraints	119
2.6.2	Relation to Green Functions ¹	125
2.7	Minimum Variance Estimation & Simultaneous Equations	126
2.7.1	The Fundamental Result	127
2.7.2	Linear Algebraic Equations	130
2.7.3	Testing After the Fact	132
2.7.4	Use of Basis Functions	134
2.7.5	Determining a Mean Value	135
2.8	Improving Recursively	138
2.9	A Recapitulation	144
2.10	Appendix 1. Maximum Likelihood	147
2.11	Appendix 2. Differential Operators and Green Functions	148
2.12	Appendix 3 Recursive Least-Squares and Gauss-Markov Solutions	149
2.13	Exercises	151
3	Extensions of Methods	161
3.1	The General Eigenvector/Eigenvalue Problem	161
3.2	Sampling	163
3.2.1	One-Dimensional Interpolation	167
3.2.2	Higher Dimensional Mapping	171
3.2.3	Mapping Derivatives	174

3.3	Inequality Constraints; Nonnegative Least Squares	174
3.4	Linear Programming	176
3.5	Empirical Orthogonal Functions	179
3.6	Kriging and Other Variants of Gauss-Markov Estimation	180
3.7	Nonlinear Problems	181
3.7.1	Total Least Squares	181
3.7.2	Method of Total Inversion	182
3.7.3	Variant Nonlinear Methods, Including Combinatorial Ones	186
4	The Time-Dependent Inverse Problem: State Estimation	189
4.1	Background	189
4.2	Basic Ideas and Notation	191
4.2.1	Models	191
4.2.2	How to Find the Matrix $\mathbf{A}(t)$	199
4.2.3	Observations and Data	201
4.3	Estimation	203
4.3.1	Model and Data Consistency	203
4.3.2	The Kalman Filter	207
4.3.3	The Smoothing Problem	218
4.3.4	Other Smoothers	229
4.4	Control and Estimation Problems	229
4.4.1	Lagrange Multipliers and Adjoints	229
4.4.2	Terminal Constraint Problem: Open Loop Control	233
4.4.3	Representers and Boundary Green Functions	239
4.4.4	The Control Riccati Equation	243
4.4.5	The Initialization Problem	244
4.5	Duality and Simplification: The Steady-State Filter and Adjoint	245
4.6	Controllability and Observability	248
4.7	Nonlinear Models	250
4.7.1	The Linearized and Extended Kalman Filter	251
4.7.2	Parameter Estimation and Adaptive Estimation	253
4.7.3	Nonlinear Adjoint Equations; Searching for Solutions	254
4.7.4	Automatic Differentiation, Linearization, and Sensitivity	256
4.7.5	Approximate Methods	261
4.8	Forward Models	262

4.9 A Summary	264
5 Time-Dependent Methods—2	273
5.1 Monte Carlo/Ensemble Methods	273
5.1.1 Ensemble Methods and Particle Filters	273
5.2 Numerical Engineering: The Search for Practicality	277
5.2.1 Meteorological Assimilation	277
5.2.2 Nudging and Objective Mapping	278
5.3 Approximate Filter/Smoother Methods	281
5.4 Reduced State Methods	285
5.5 Uncertainty in Lagrange Multiplier Method	287
5.6 Non-normal Systems	287
5.6.1 POPs and Optimal Modes	289
5.7 Adaptive Problems	290
II Applications	293
6 Applications to Steady Problems	295
6.1 Steady-State Tracer Distributions	296
6.2 The Steady Ocean Circulation Inverse Problem	299
6.2.1 Equations of Motion	299
6.2.2 Geostrophy	299
6.2.3 Integral Version	304
6.2.4 Discrete Version	304
6.2.5 A Specific Example	306
6.2.6 Solution by SVD	311
6.2.7 Solution by Gauss-Markov Estimate	322
6.2.8 Adding Further Properties	322
6.3 Property Fluxes	329
6.4 Application to Real Oceanographic Problems	331
6.4.1 Regional Applications	331
6.4.2 The Columnar Result	339
6.4.3 Global-Scale Applicationsf	340
6.4.4 Error Estimates	342
6.4.5 Finite-Difference Models	346

CONTENTS

ix

6.5	Linear Programming Solutions	347
6.6	The β -Spiral and Variant Methods	349
6.6.1	The β -Spiral	349
6.7	Alleged Failure of Inverse Methods	354
6.8	Applications of EOFs (Singular Vectors)	354
6.9	Non-linear Problems	356
7	Applications to Time-Dependent Fluid Problems	361
7.1	Time Dependent Tracers	362
7.2	Global Ocean States by Lagrange Multiplier Methods	362
7.3	Global Ocean States by Sequential Methods	370
7.4	Tomographic Applications	372
7.5	Appendix	373