8. Circular Convolution

There is one potentially puzzling feature of convolution for discrete sequences. Suppose one has $f_m \neq 0, = 0, 1, 2,$ and is zero otherwise, and that $g_m \neq 0, m = 0, 1, 2,$ and is zero otherwise. Then $h = f * g$ is,

$$[h_0, h_1, h_2, h_3, h_4, h_5] = [f_0 g_0, f_0 g_1 + f_1 g_0, f_0 g_2 + f_1 g_1 + f_2 g_0, f_1 g_2 + f_2 g_0, f_0 g_2], \tag{8.1}$$

that is, is non-zero for 5 elements. But the product $\hat{f}(z)\,\hat{g}(z)$ is the Fourier transform of only a 3-term non-zero sequence. How can the two results be consistent? Note that $\hat{f}(z), \hat{g}(z)$ are Fourier transforms of two sequences which are numerically indistinguishable from periodic ones with period 2. Thus their product must also be a Fourier transform of a sequence indistinguishable from periodic with period 2.

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$\hat{f}(z)\,\hat{g}(z)$ is the Fourier transform of the convolution of two periodic sequences f_m, g_m, not the ones in Eq. (8.1) that we have treated as being zero outside their region of definition. $\mathcal{Z}^{-1}\left(\hat{f}(z)\,\hat{g}(z)\right)$ is the convolutionof two periodic sequences, and which have "wrapped around" on each other—giving rise to their description as "circular convolution". To render circular convolution identical to Eq. (8.1), one should pad f_m, g_m with enough zeros that their lengths are identical to that of h_m before forming $\hat{f}(z)\,\hat{g}(z)$.

In a typical situation however, f_m might be a simple filter, perhaps of length 10, and g_m might be a set of observations, perhaps of length 10,000. If one simply drops the five points on each end for which the convolution overlaps the zeros "off-the-ends", then the two results are virtually identical. An extended discussion of this problem can be found in Press et al. (1992, Section 12.4).