

## 2. Geometric Interpretation

There is a geometric interpretation of the normal equations. Let us define vectors (of nominally infinite length) as

$$\mathbf{x}_r = \left[ \dots x_{r-1}, x_{r-1}, \underset{\uparrow}{x_r}, x_{r+1}, x_{r+2}, \dots \right]^T \quad (2.1)$$

$$\boldsymbol{\theta}_r = \left[ \dots \theta_{r-1}, \theta_{r-1}, \underset{\uparrow}{\theta_r}, \theta_{r+1}, \theta_{r+2}, \dots \right]^T \quad (2.2)$$

where the arrow denotes the time origin (these vectors are made up of the elements of the time series, “slid over” so that element  $r$  lies at the time origin). Define the inner (dot) products of these vectors in the usual way, ignoring any worries about convergence of infinite sums. Let us attempt to expand vector  $\mathbf{x}_r$  in terms of  $M$ -past vectors:

$$\mathbf{x}_r = a_1 \mathbf{x}_{r-1} + a_2 \mathbf{x}_{r-2} + \dots + a_M \mathbf{x}_{r-M} + \varepsilon_r \quad (2.3)$$

where  $\varepsilon_r$  is the residual of the fit. Best fits are found by making the residuals orthogonal to the expansion vectors:

$$\mathbf{x}_{r-i}^T (\mathbf{x}_r - a_1 \mathbf{x}_{r-1} - a_2 \mathbf{x}_{r-2} - \dots - a_M \mathbf{x}_{r-M}) = 0, 1 \leq i \leq M \quad (2.4)$$

which produces, after dividing all equations by  $N$ , and taking the limit as  $N \rightarrow \infty$ ,

$$a_1 R(0) + a_2 R(1) + \dots + a_M R(M-1) = R(1) \quad (2.5)$$

$$a_1 R(1) + a_2 R(0) + \dots + R(M-2) = R(2) \quad (2.6)$$

$$\dots, \quad (2.7)$$

that is precisely the Yule-Walker equations, but with the theoretical values of  $R$  replacing the estimated ones. One can build this view up into a complete vector space theory of time series analysis. By using the actual finite length vectors as approximations to the infinite length ones, one connects this theoretical construct to the one used in practice. Evidently this form of time series analysis is equivalent to the study of the expansion of a vector in a set of (generally non-orthogonal) vectors.