

1. Fourier Transforms and Delta Functions

“Time” is the physical variable, written as t , although it may well be a spatial coordinate. Let $x(t), y(t)$, etc. be real, continuous, well-behaved functions. The meaning of “well-behaved” is not so-clear. For Fourier transform purposes, it classically meant among other requirements, that

$$\int_{-\infty}^{\infty} |x(t)|^2 < \infty. \quad (1.1)$$

Unfortunately such useful functions as $x(t) = \sin(2\pi t/T)$, or

$$\begin{aligned} x(t) &= H(t) = 0, t < 0 \\ &= 1, t \geq 0 \end{aligned} \quad (1.2)$$

are excluded (the latter is the unit step or Heaviside function). We succeed in including these and other useful functions by admitting the existence and utility of Dirac δ -functions. (A textbook would specifically exclude functions like $\sin(1/t)$. In general, such functions do not appear as physical signals and I will rarely bother to mention the rigorous mathematical restrictions on the various results.)

The Fourier transform of $x(t)$ will be written as

$$\mathcal{F}(x(t)) \equiv \hat{x}(s) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i s t} dt. \quad (1.3)$$

It is often true that

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(s) e^{2\pi i s t} ds \equiv \mathcal{F}^{-1}(\hat{x}(s)). \quad (1.4)$$

Other conventions exist, using radian frequency ($\omega = 2\pi s$), and/or reversing the signs in the exponents of (1.3, 1.4). All are equivalent (I am following Bracewell’s convention).

Exercise. The Fourier transform pair (1.3, 1.4) is written in complex form. Re-write it as cosine and sine transforms where all operations are real. Discuss the behavior of $\hat{x}(s)$ when $x(t)$ is an even and odd function of time.

Define $\delta(t)$ such that

$$x(t_0) = \int_{-\infty}^{\infty} x(t) \delta(t_0 - t) dt \quad (1.5)$$

It follows immediately that

$$\mathcal{F}(\delta(t)) = 1 \quad (1.6)$$

and therefore that

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} ds = \int_{-\infty}^{\infty} \cos(2\pi s t) ds. \quad (1.7)$$

Notice that the δ -function has units; Eq. (1.5) implies that the units of $\delta(t)$ are $1/t$ so that the equation works dimensionally.

Definition. A “sample” value of $x(t)$ is $x(t_m)$, the value at the specific time $t = t_m$.

We can write, in seemingly cumbersome fashion, the sample value as

$$x(t_m) = \int_{-\infty}^{\infty} x(t) \delta(t_m - t) dt \quad (1.8)$$

This expression proves surprisingly useful.

Exercise. With $x(t)$ real, show that

$$\hat{x}(-s) = \hat{x}(s)^* \quad (1.9)$$

where $*$ denotes the complex conjugate.

Exercise. a is a constant. Show that

$$\mathcal{F}(x(at)) = \frac{1}{|a|} \hat{x}\left(\frac{s}{a}\right). \quad (1.10)$$

This is the scaling theorem.

Exercise. Show that

$$\mathcal{F}(x(t-a)) = e^{-2\pi i a s} \hat{x}(s). \quad (1.11)$$

(shift theorem).

Exercise. Show that

$$\mathcal{F}\left(\frac{dx(t)}{dt}\right) = 2\pi i s \hat{x}(s). \quad (1.12)$$

(differentiation theorem).

Exercise. Show that

$$\mathcal{F}(x(-t)) = \hat{x}(s)^* \quad (1.13)$$

(time-reversal theorem)

Exercise. Find the Fourier transforms of $\cos 2\pi s_0 t$ and $\sin 2\pi s_0 t$. Sketch and describe them in terms of real, imaginary, even, odd properties.

Exercise. Show that if $x(t) = x(-t)$, that is, x is an “even-function”, then

$$\hat{x}(s) = \hat{x}(-s), \quad (1.14)$$

and that it is real. Show that if $x(t) = -x(-t)$, (an “odd-function”), then

$$\hat{x}(s) = \hat{x}(-s)^*, \quad (1.15)$$

and it is pure imaginary.

Note that any function can be written as the sum of an even and odd-function

$$\begin{aligned} x(t) &= x_e(t) + x_o(t) \\ x_e(t) &= \frac{1}{2}(x(t) + x(-t)), \quad x_o(t) = \frac{1}{2}(x(t) - x(-t)). \end{aligned} \quad (1.16)$$

Thus,

$$\hat{x}(s) = \hat{x}_e(s) + \hat{x}_o(s). \quad (1.17)$$

There are two fundamental theorems in this subject. One is the proof that the transform pair (1.3,1.4) exists. The second is the so-called convolution theorem. Define

$$h(t) = \int_{-\infty}^{\infty} f(t') g(t-t') dt' \quad (1.18)$$

where $h(t)$ is said to be the “convolution” of f with g . The convolution theorem asserts:

$$\hat{h}(s) = \hat{f}(s) \hat{g}(s). \quad (1.19)$$

Convolution is so common that one often writes $h = f * g$. Note that it follows immediately that

$$f * g = g * f. \quad (1.20)$$

Exercise. Prove that (1.19) follows from (1.18) and the definition of the Fourier transform. What is the Fourier transform of $x(t)y(t)$?

Exercise. Show that if,

$$h(t) = \int_{-\infty}^{\infty} f(t') g(t+t') dt' \quad (1.21)$$

then

$$\hat{h}(s) = \hat{f}(s)^* \hat{g}(s) \quad (1.22)$$

$h(t)$ is said to be the “cross-correlation” of f and g , written here as $h = f \otimes g$. Note that $f \otimes g \neq g \otimes f$.

If $g = f$, then (1.21) is called the “autocorrelation” of f (a better name is “autocovariance”, but the terminology is hard to displace), and its Fourier transform is,

$$\hat{h}(s) = |\hat{f}(s)|^2 \quad (1.23)$$

and is called the “power spectrum” of $f(t)$.

Exercise: Find the Fourier transform and power spectrum of

$$\Pi(t) = \begin{cases} 1, & |t| \leq 1/2 \\ 0, & |t| > 1/2. \end{cases} \quad (1.24)$$

Now do the same, using the scaling theorem, for $\Pi(t/T)$. Draw a picture of the power spectrum.

One of the fundamental Fourier transform relations is the Parseval (sometimes, Rayleigh) relation:

$$\int_{-\infty}^{\infty} x(t)^2 dt = \int_{-\infty}^{\infty} |\hat{x}(s)|^2 ds. \quad (1.25)$$

1. FREQUENCY DOMAIN FORMULATION

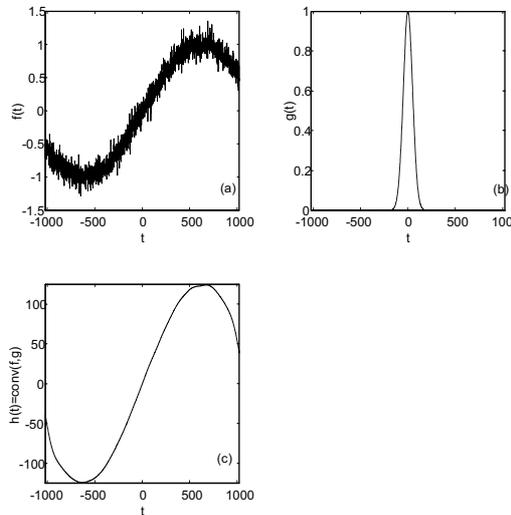


FIGURE 1. An effect of convolution is for a “smooth” function to reduce the high frequency oscillations in the less smooth function. Here a noisy curve (a) is convolved with a smoother curve (b) to produce the result in (c), where the raggedness of the the noisy function has been greatly reduced. (Technically here, the function in (c) is a low-pass filter. No normalization has been imposed however; consider the magnitude of (c) compared to (a).)

Exercise. Using the convolution theorem, prove (1.25).

Exercise. Using the definition of the δ -function, and the differentiation theorem, find the Fourier transform of the Heaviside function $H(t)$. Now by the same procedure, find the Fourier transform of the sign function,

$$\text{signum}(t) = \text{sgn}(t) = \begin{cases} -1, & t < 0 \\ 1, & t > 0 \end{cases}, \quad (1.26)$$

and compare the two answers. Can both be correct? Explain the problem. (Hint: When using the differentiation theorem to deduce the Fourier transform of an integral of another function, one must be aware of integration constants, and in particular that functions such as $s\delta(s) = 0$ can always be added to a result without changing its value.) Solution:

$$\mathcal{F}(\text{sgn}(t)) = \frac{-i}{\pi s}. \quad (1.27)$$

Often one of the functions $f(t)$, $g(t)$ is a long “wiggly” curve, (say) $g(t)$ and the other, $f(t)$ is comparatively simple and compact, for example as shown in Fig. 1 The act of convolution in this situation tends to subdue the oscillations in and other structures in $g(t)$. In this situation $f(t)$ is usually called a “filter”, although which is designated as the filter is clearly an arbitrary choice. Filters exist for and are designed for, a very wide range of purposes. Sometimes one wishes to change the frequency

content of $g(t)$, leading to the notion of high-pass, low-pass, band-pass and band-rejection filters. Other filters are used for prediction, noise suppression, signal extraction, and interpolation.

Exercise. Define the “mean” of a function to be,

$$m = \int_{-\infty}^{\infty} f(t) dt, \quad (1.28)$$

and its “variance”,

$$(\Delta t)^2 = \int_{-\infty}^{\infty} (t - m)^2 f(t) dt. \quad (1.29)$$

Show that

$$\Delta t \Delta s \geq \frac{1}{4\pi}. \quad (1.30)$$

This last equation is known as the “uncertainty principle” and occurs in quantum mechanics as the Heisenberg Uncertainty Principle, with momentum and position being the corresponding Fourier transform domains. You will need the Parseval Relation, the differentiation theorem, and the Schwarz Inequality:

$$\left| \int_{-\infty}^{\infty} f(t) g(t) dt \right|^2 \leq \int_{-\infty}^{\infty} |f(t)|^2 dt \int_{-\infty}^{\infty} |g(t)|^2 dt. \quad (1.31)$$

The uncertainty principle tells us that a narrow function must have a broad Fourier transform, and vice-versa with “broad” being defined as being large enough to satisfy the inequality. Compare it to the scaling theorem. Can you find a function for which the inequality is actually equality?

1.1. The Special Role of Sinusoids. One might legitimately inquire as to why there is such a specific focus on the sinusoidal functions in the analysis of time series? There are, after all, many other possible basis functions (Bessel, Legendre, etc.). One important motivation is their role as the eigenfunctions of extremely general linear, time-independent systems. Define a linear system as an operator $\mathcal{L}(\cdot)$ operating on any input, $x(t)$. \mathcal{L} can be a physical “black-box” (an electrical circuit, a pendulum, etc.), and/or can be described via a differential, integral or finite difference, operator. \mathcal{L} operates on its input to produce an output:

$$y(t) = \mathcal{L}(x(t), t). \quad (1.32)$$

It is “time-independent” if \mathcal{L} does not depend explicitly on t , and it is linear if

$$\mathcal{L}(ax(t) + w(t), t) = a\mathcal{L}(x(t), t) + \mathcal{L}(w(t), t) \quad (1.33)$$

for any constant a . It is “causal” if for $x(t) = 0, t < t_0$, $\mathcal{L}(x(t)) = 0, t < t_0$. That is, there is no response prior to a disturbance (most physical systems satisfy causality).

Consider a general time-invariant linear system, subject to a complex periodic input:

$$y(t) = \mathcal{L}(e^{2\pi i s_0 t}). \quad (1.34)$$

Suppose we introduce a time shift,

$$y(t + t_0) = \mathcal{L}(e^{2\pi i s_0 (t+t_0)}). \quad (1.35)$$

Now set $t = 0$, and

$$y(t_0) = \mathcal{L}(e^{2\pi i s_0 t_0}) = e^{2\pi i s_0 t_0} \mathcal{L}(e^{2\pi i_0 s t=0}) = e^{2\pi i s_0 t_0} \mathcal{L}(1). \quad (1.36)$$

Now $\mathcal{L}(1)$ is a constant (generally complex). Thus (1.36) tells us that for an input function $e^{2\pi i s_0 t_0}$, with both s_0, t_0 completely arbitrary, the output must be another pure sinusoid—at exactly the same period—subject only to a modification in amplitude and phase. This result is a direct consequence of the linearity and time-independence assumptions. Eq. (1.36) is also a statement that any such exponential is thereby an eigenfunction of \mathcal{L} , with eigenvalue $\mathcal{L}(1)$. It is a very general result that one can reconstruct arbitrary linear operators from their eigenvalues and eigenfunctions, and hence the privileged role of sinusoids; in the present case, that reduces to recognizing that the Fourier transform of $y(t_0)$ would be that of \mathcal{L} which would thereby be fully determined. (One can carry out the operations leading to (1.36) using real sines and cosines. The algebra is more cumbersome.)

1.2. Causal Functions and Hilbert Transforms. Functions that vanish before $t = 0$ are said to be “causal”. By a simple shift in origin, any function which vanishes for $t < t_0$ can be reduced to a causal one, and it suffices to consider only the special case, $t_0 = 0$. The reason for the importance of these functions is that most physical systems obey a causality requirement that they should not produce any output, before there is an input. (If a mass-spring oscillator is at rest, and then is disturbed, one does not expect to see it move before the disturbance occurs.) Causality emerges as a major requirement for functions which are used to do prediction—they cannot operate on future observations, which do not yet exist, but only on the observed past.

Consider therefore, any function $x(t) = 0, t < 0$. Write it as the sum of an even and odd-function,

$$x(t) = \begin{cases} x_e(t) + x_o(t) = \frac{1}{2}(x(t) + x(-t)) + \frac{1}{2}(x(t) - x(-t)) \\ = 0, t < 0, \end{cases} \quad (1.37)$$

but neither $x_e(t)$, nor $x_o(t)$ vanishes for $t < 0$, only their sum. It follows from (1.37) that

$$x_o(t) = \text{sgn}(t) x_e(t) \quad (1.38)$$

and that

$$x(t) = (1 + \text{sgn}(t)) x_e(t). \quad (1.39)$$

Fourier transforming (1.39), and using the convolution theorem, we have

$$\hat{x}(s) = \hat{x}_e(s) + \frac{-i}{\pi s} * \hat{x}_e(s) \quad (1.40)$$

using the Fourier transform for $\text{sgn}(t)$.

Because $\hat{x}_e(s)$ is real, the imaginary part of $\hat{x}(s)$ must be

$$\text{Im}(\hat{x}(s)) = \hat{x}_o(s) = \frac{-1}{\pi s} * \hat{x}_e(s) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{x}_e(s')}{s - s'} ds'. \quad (1.41)$$

Re-writing (1.39) in the obvious way in terms of $x_o(t)$, we can similarly show,

$$\hat{x}_e(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{x}_o(s')}{s-s'} ds'. \quad (1.42)$$

Eqs. (1.41, 1.42) are a pair, called Hilbert transforms. Causal functions thus have intimately connected real and imaginary parts of their Fourier transforms; knowledge of one determines the other. These relationships are of great theoretical and practical importance. An oceanographic application is discussed in Wunsch (1972).

The Hilbert transform can be applied in the time domain to a function $x(t)$, whether causal or not. Here we follow Bendat and Piersol (1986, Chapter 13). Define

$$x^H(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(t')}{t-t'} dt' \quad (1.43)$$

and $x(t)$ can be recovered from $x^H(t)$ by the inverse Hilbert transform (1.42). Eq. (1.43) is the convolution

$$x^H(t) = x(t) * \frac{1}{\pi t} \quad (1.44)$$

and by the convolution theorem,

$$\hat{x}^H(s) = \hat{x}(s) (-i \operatorname{sgn}(s)) \quad (1.45)$$

using the Fourier transform of the signum function. The last expression can be re-written as

$$\hat{x}^H(s) = \hat{x}(s) \begin{cases} \exp(-i\pi/2), & s < 0 \\ \exp(i\pi/2), & s > 0 \end{cases}, \quad (1.46)$$

that is, the Hilbert transform in time is equivalent to phase shifting the Fourier transform of $x(t)$ by $\pi/2$ for positive frequencies, and by $-\pi/2$ for negative ones. Thus $x^H(t)$ has the same frequency content of $x(t)$, but is phase shifted by 90° . It comes as no surprise therefore, that if e.g., $x(t) = \cos(2\pi s_0 t)$, then $x^H(t) = \sin(2\pi s_0 t)$. Although we do not pursue it here (see Bendat and Piersol, 1986), this feature of Hilbert transformation leads to the idea of an “analytic signal”,

$$y(t) = x(t) + ix^H(t) \quad (1.47)$$

which proves useful in defining an “instantaneous frequency”, and in studying the behavior of wave propagation including the idea (taken up much later) of complex empirical orthogonal functions.

Writing the inverse transform of a causal function,

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(s) e^{2\pi i s t} ds, \quad (1.48)$$

one might, if $\hat{x}(s)$ is suitably behaved, attempt to evaluate this transform by Cauchy’s theorem, as

$$x(t) = \begin{cases} 2\pi i \sum (\text{residues of the lower half-}s\text{-plane, }), & t < 0 \\ 2\pi i \sum (\text{residues of the upper half-}s\text{-plane, }), & t > 0 \end{cases} \quad (1.49)$$

Since the first expression must vanish, if $\hat{x}(s)$ is a rational function, it cannot have any poles in the lower-half- s -plane; this conclusion leads immediately so-called Wiener filter theory, and the use of Wiener-Hopf methods.

1.3. Asymptotics. The gain of insight into the connections between a function and its Fourier transform, and thus developing intuition, is a very powerful aid to interpreting the real world. The scaling theorem, and its first-cousin, the uncertainty principle, are part of that understanding. Another useful piece of information concerns the behavior of the Fourier transform as $|s| \rightarrow \infty$. The classical result is the Riemann-Lebesgue Lemma. We can write

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i s t} dt. \quad (1.50)$$

where here, $f(t)$ is assumed to satisfy the classical conditions for existence of the Fourier transform pair. Let $t' = t - 1/(2s)$, (note the units are correct) then by a simple change of variables rule,

$$\hat{f}(s) = - \int_{-\infty}^{\infty} f\left(t' + \frac{1}{2s}\right) e^{-2\pi i s t'} dt' \quad (1.51)$$

($\exp(-i\pi) = -1$) and taking the average of these last two expressions, we have,

$$\begin{aligned} |\hat{f}(s)| &= \left| \frac{1}{2} \int_{-\infty}^{\infty} f(t) e^{-2\pi i s t} dt - \frac{1}{2} \int_{-\infty}^{\infty} f\left(t + \frac{1}{2s}\right) e^{-2\pi i s t} dt \right| \\ &\leq \frac{1}{2} \int_{-\infty}^{\infty} \left| f(t) - f\left(t + \frac{1}{2s}\right) \right| dt \rightarrow 0, \text{ as } s \rightarrow \infty \end{aligned} \quad (1.52)$$

because the difference between the two functions becomes arbitrarily small with increasing $|s|$.

This result tells us that for classical functions, we are assured that for sufficiently large $|s|$ the Fourier transform will go to zero. It doesn't however, tell us how fast it does go to zero. A general theory is provided by Lighthill (1958), which he then builds into a complete analysis system for asymptotic evaluation. He does this essentially by noting that functions such as $H(t)$ have Fourier transforms which for large $|s|$ are dominated by the contribution from the discontinuity in the first derivative, that is, for large s , $H(s) \rightarrow 1/s$ (compare to *signum*(t)). Consideration of functions whose first derivatives are continuous, but whose second derivatives are discontinuous, shows that they behave as $1/s^2$ for large $|s|$; in general if the n -th derivative is the first discontinuous one, then the function behaves asymptotically as $1/|s|^n$. These are both handy rules for what happens and useful for evaluating integrals at large s (or large distances if one is going from Fourier to physical space). Note that even the δ -function fits: its 0-th derivative is discontinuous (that is, the function itself), and its asymptotic behavior is $1/s^0 = \text{constant}$; it does not decay at all as it violates the requirements of the Riemann-Lebesgue lemma.