15. Spectral Peaks

Pure sinusoids are very rare in nature, generally being associated with periodic astronomical phenomena such as tides, or the Milankovitch forcing in solar insolation. Thus apart from phenomena associated with these forcings, deterministic sinusoids are primarily of mathematical interest. To the extent that one suspects a pure "tone" in a record not associated with an astronomical forcing, it would be a most unusual, not-to-say startling, discovery. (These are called "line spectra".) If you encounter a paper claiming to see pure frequency lines at anything other than at the period of an astronomical phenomenon, it's a good bet that the author doesn't know what he is doing. (Many people like to believe that the world is periodic; mostly it isn't.)

But because both tides and Milankovitch responses are the subject of intense interest in many fields, it is worth a few comments about line spectra. We have already seen that unlike a stochastic process, the

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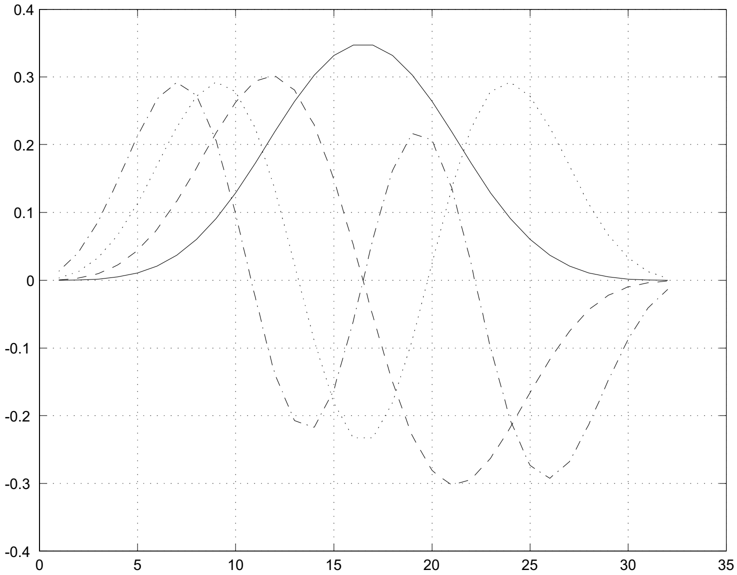

FIGURE 21. First 4 discretized prolate spheroidal wave functions (also known as Slepian sequences) used in the multitaper method for a data duration of 32. Methods for computing these functions are described in detail by Percival and Walden (1993) and they were found here using a MATLAB toolbox function. Notice that as the order increases, greater weight is given to data near the ends of the observations.

Fourier coefficients of a pure sinusoid do not diminish as the record length, N, increases (alternatively, the Fourier transform value increases with N, while for the stochastic process they remain fixed in rms amplitude). This behavior produces a simple test of the presence of a pure sinusoid: double (or halve) the record length, and determine whether the Fourier coefficient remains the same or changes.

Much more common in nature are narrow-band peaks, which represent a relative excess of energy, but which is stochastic, and not a deterministic sinusoid. A prominent example is the peak in the current meter spectra (Fig.14) associated with the Coriolis frequency). The ENSO peak in the Southern Oscillation Index (Wunsch, 1999) is another example, and many others exist. None of these phenomena are represented by line spectra. Rather they are what is sometimes called "narrow-band" random processes. It proves convenient to have a common measure of the sharpness of a peak, and this measure is provided by what electrical engineers call Q (for "quality factor" associated with a degree of resonance). A damped mass-spring oscillator, forced by white noise, and satisfying an equation like

$$m\frac{d^2x}{dt^2}+r\frac{dx}{dt}+kx=\theta\left(t\right) \tag{15.1}$$

will display an energy peak near frequency $s=\left(2\pi\right)^{-1}\sqrt{k/m,}.$ as in Fig. 22. The sharpness of the peak depends upon the value of r. Exactly the same behavior is found in almost any linear oscillator (e.g., an

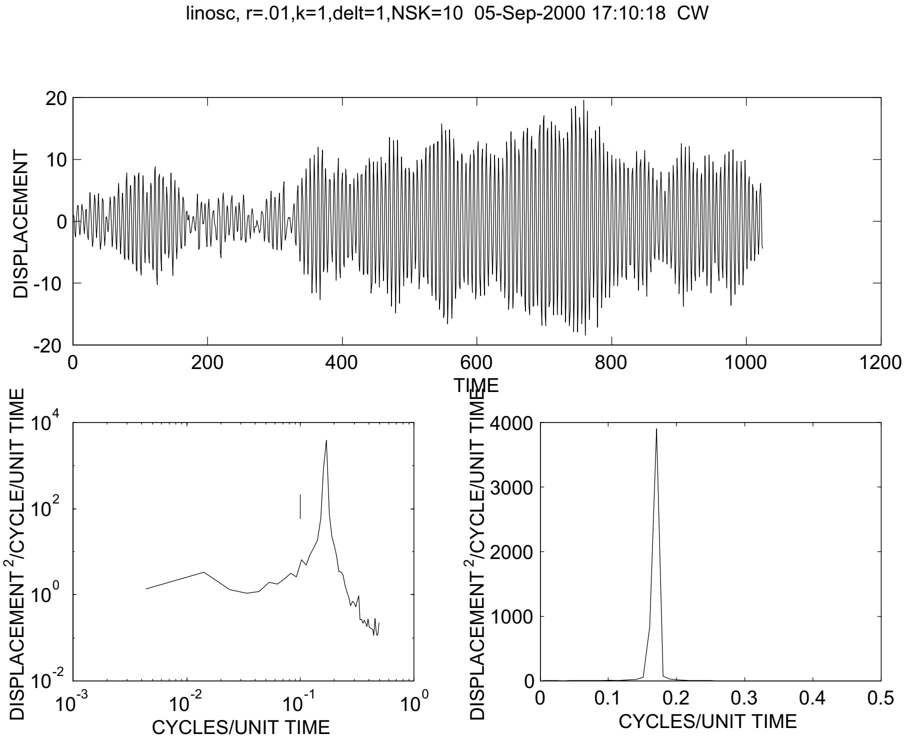

Figure 22. (Top) Time series of displacement of a simple mass spring oscillator, driven by white noise, and computed numerically such that $r/m = 0.1, k/m = 1$ with $\Delta t = 1$. Lower left and right panels are the estimated power density spectrum plotted differently. The Q here exceeds about 20.

organ pipe, or an $L - C$ electrical circuit). Peak width is measured in terms of

$$Q = \frac{s_0}{\Delta s}, \tag{15.2}$$

(e.g., *Jackson,* 1975). Here s_0 is the circular frequency of the peak center and Δs is defined as the bandwidth of the peak at its half-power points. For linear systems such as (15.1), it is an easy matter to show that an equivalent definition is,

$$Q = \frac{2\pi E}{< dE/dt >} \tag{15.3}$$

where here E is the peak energy stored in the system, and $< dE/dt >$ is the mean rate of energy dissipated over one cycle. It follows that for (15.1),

$$Q = \frac{2\pi s_0}{r} \tag{15.4}$$

(Jackson, 1975; Munk and Macdonald, 1960, p. 22). As r diminishes, the resonance is greater and greater, $\Delta s \rightarrow 0, < dE/dt > \rightarrow 0$, and $Q \rightarrow \infty$, the resonance becoming perfect.

Exercise. Write (15.1) in discrete form; calculate numerical x_m for white noise forcing, and show that the power density spectrum of the result is consistent with the three definitions of Q.

Exercise. For the parameters given in the caption of Fig. 22, calculate the value of Q.

Values of Q for the ocean tend to be in the range of $1-20$ (see, e.g., Luther, 1982). The lowest free elastic oscillation of the earth (first radial mode) has a Q approaching 10,000 (the earth rings like a bell for months after a large earthquake), but this response is extremely unusual in nature and such a mode may well be thought of as an astronomical one.

A Practical Point

The various normalizations employed for power densities and related estimates can be confusing if one, for example, wishes to compute the rms amplitude of the motion in some frequency range, e.g., that corresponding to a local peak. Much of the confusion can be evaded by employing the Parseval relationship (2.4). First compute the record variance, $\tilde{\sigma}^2$, and form the accumulating sum $\tilde{\Phi}^d(s_n) = \sum_{k=1}^{n}\left(a_k^2+b_k^2\right), n \leq [N/2]$, assuming negligible energy in the mean. Then the fraction of the power lying between any two frequencies, n, n', must be $\left(\tilde{\Phi}^d(s_n)-\tilde{\Phi}^d(s'_n)\right)/\tilde{\Phi}^d\left(s_{[N/2]}\right)$; the root-mean-square amplitude corresponding to that energy is

$$\left(\sqrt{\left(\tilde{\Phi}^d(s_n)-\tilde{\Phi}^d(s'_n)\right)/\tilde{\Phi}^d\left(s_{[N/2]}\right)}\right)\tilde{\sigma}/\sqrt{2}, \tag{15.5}$$

and the normalizations used for $\tilde{\Phi}$ drop out.