3. The Sampling Theorem

We have seen that a time-limited function can be reconstructed from its Fourier coefficients. The reader will probably have noticed that there is symmetry between frequency and time domains. That is to say, apart from the assignment of the sign of the exponent of $\exp(2\pi ist)$, the s and t domains are essentially equivalent. For many purposes, it is helpful to use not t, s with their physical connotations, but abstract symbols like q, r. Taking the lead from this inference, let us interchange the t, s domains in the equations (2.6, 2.13), making the substitutions $t \to s, s \to t, T \to 1/\Delta t, \hat{x}(s) \to x(t)$.. We then have,

$$\hat{x}(s) = 0, s \geq 1/2\Delta t \tag{3.1}$$

$$\begin{aligned} x(t) &= \sum_{m=-\infty}^{\infty} x(m\Delta t) \frac{\sin(\pi(m - t/\Delta t))}{\pi(m - t/\Delta t)} \\ &= \sum_{m=-\infty}^{\infty} x(m\Delta t) \frac{\sin((\pi/\Delta t)(t - m\Delta t))}{(\pi/\Delta t)(t - m\Delta t)}. \end{aligned} \tag{3.2}$$

This result asserts that a function *bandlimited* to the frequency interval $|s| \leq 1/2\Delta t$ (meaning that its Fourier transform vanishes for all frequencies outside of this *baseband*) can be perfectly reconstructed by samples of the function at the times $m\Delta t$. This result (3.1,3.2) is the famous Shannon sampling theorem. As such, it is an *interpolation statement.* It can also be regarded as a statement of information content: all of the information about the bandlimited continuous time series is contained in the samples. This result is actually a remarkable one, as it asserts that a continuous function with an uncountable infinity of points can be reconstructed from a countable infinity of values.

Although one should never use (3.2) to interpolate data in practice (although so-called *sinc* methods are used to do numerical integration of analytically-defined functions), the implications of this rule are very important and can be stated in a variety of ways. In particular, let us write a general bandlimiting form:

$$\hat{x}(s) = 0, s \geq s_c \tag{3.3}$$

If (3.3) is valid, it *suffices* to sample the function at *uniform* time intervals $\Delta t \leq 1/2s_c$ (Eq. 3.1 is clearly then satisfied.).

Exercise. Let $\Delta t = 1$. $x(t)$ is measured at all times, and found to vanish, except for $t = m = 0, 1, 2, 3$ and the values are $[1, 2, -1, -1]$. Calculate the values of $x(t)$ at intervals $\Delta t/10$ from $-5 \leq t \leq 5$ and plot it. Find the Fourier transform of $x(t)$.

The consequence of the sampling theorem for discrete observations in time is that there is no purpose in calculating the Fourier transform for frequencies larger in magnitude than $1/(2\Delta t)$. Coupled with the result for time-limited functions, we conclude that *all of the information about a finite sequence of N observations at intervals Δt and of duration, $(N-1)\Delta t$ is contained in the baseband $|s| \leq 1/2\Delta t$, at frequencies $s_n = n/(N\Delta t)$.*

There is a theorem (owing to Paley and Wiener) that a time-limited function cannot be band-limited, and vice-versa. One infers that a truly time-limited function must have a Fourier transform with non-zero values extending to arbitrarily high frequencies, s. If such a function is sampled, then some degree of aliasing is inevitable. For a truly band-limited function, one makes the required interchange to show that it must actually extend with finite values to $t = \pm\infty$. Some degree of aliasing of real signals is therefore inevitable. Nonetheless, such aliasing can usually be rendered arbitrarily small and harmless; the need to be vigilant is, however, clear.

3.1. Tapering, Leakage, Etc. Suppose we have a continuous cosine $x(t) = \cos(2\pi p_1 t/T_1)$. Then the true Fourier transform is

$$\hat{x}(s) = \frac{1}{2}\left\{\delta(s - p_1) + \delta(s + p_1)\right\}. \tag{3.4}$$

If it is observed (continuously) over the interval $-T/2 \leq t \leq T/2$, then we have the Fourier transform of

$$x_\Pi(t) = x(t)\,\Pi(t/T) \tag{3.5}$$

and which is found immediately to be

$$\hat{x}_\Pi(s) = \frac{T}{2}\left\{\frac{\sin(\pi T(s - p_1))}{(\pi T(s - p_1))} + \frac{\sin(\pi T(s + p_1))}{(\pi T(s + p_1))}\right\} \tag{3.6}$$

The function

$$\mathit{sinc(Ts)} = \sin(\pi T s)/(\pi T s), \tag{3.7}$$

plotted in Fig. (2), is ubiquitous in time series analysis and worth some study. Note that in (3.6) there is a "main-lobe" of width $2/T$ (defined by the zero crossings) and with amplitude maximum T. To each side of the main lobe, there is an infinite set of diminishing "sidelobes" of width $1/T$ between zero crossings. Let us suppose that p_1 in (3.4) is chosen to be one of the special frequencies $s_n = n/T, T = N\Delta t$, in particular, $p_1 = p/T$. Then (3.6) is a sum of two *sinc* functions centered at $s_p = \pm p/T$. A very important feature is that each of these functions vanishes identically at all other special frequencies $s_n, n \neq p$. If

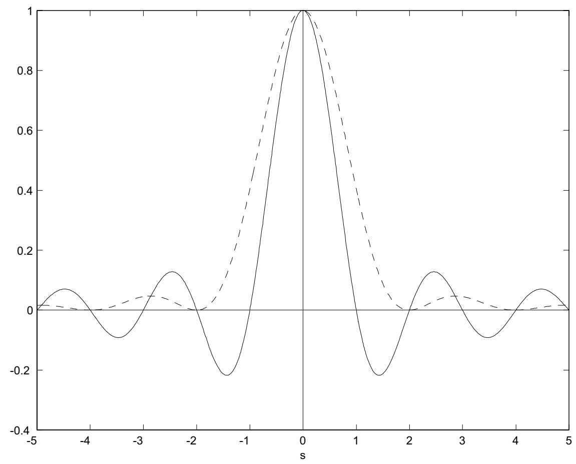

FIGURE 2. The function, $sinc(sT) = \sin(\pi sT) / (\pi sT)$, (solid line) which is the Fourier transform of a pure exponential centered at that corresponding frequency. Here $T = 1$. Notice that the function crosses zero whenever $s = m$, which corresponds to the Fourier frequency separation. The main lobe has width 2, while successor lobes have width 1, with a decay rate only as fast as $1/|s|$. The function $sinc^2(s/2)$ (dotted line) decays as $1/|s|^2$, but its main lobe appears, by the scaling theorem, with twice the width of that of the $sinc(s)$ function.

we confine ourselves, as the inferences of the previous section imply, to computing the Fourier transform at only these special frequencies, we would see only a large value T at $s = s_p$ and zero at every other such frequency. (Note that if we convert to Fourier coefficients by division by $1/T$, we obtain the proper values.) The Fourier transform does not vanish for the continuum of frequencies $s \neq s_n$, but it could be obtained from the sampling theorem.

Now suppose that the cosine is no longer a Fourier harmonic of the record length. Then computation of the Fourier transform at s_n no longer produces a zero value; rather one obtains a finite value from (3.6). In particular, if p_1 lies halfway between two Fourier harmonics, $n/T \leq p_1 \leq (n+1)/T$, $|\hat{x}(s_n)|, |\hat{x}(s_{n+1})|$ will be approximately equal, and the absolute value of the remaining Fourier coefficients will diminish roughly as $1/|n-m|$. The words "approximately" and "roughly" are employed because there is another *sinc* function at the corresponding negative frequencies, which generates finite values in the positive half of the $s-$axis. The analyst will not be able to distinguish the result (a single pure Fourier frequency *in between* s_n, s_{n+1}) from the possibility that there are *two* pure frequencies present at s_n, s_{n+1}. Thus we have what is sometimes called "Rayleigh's criterion": that to separate, or "resolve" two pure sinusoids, at frequencies p_1, p_2, their frequencies must differ by

$$|p_1 - p_2| \geq \frac{1}{T},. \tag{3.8}$$

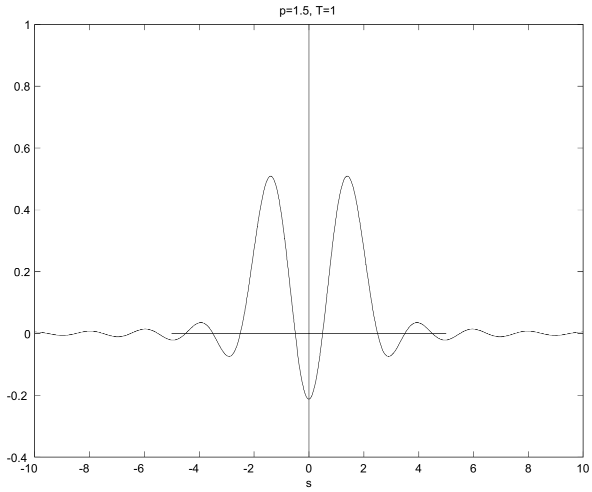

FIGURE 3. Interference pattern from a cosine, showing how contributions from positive and negative frequencies add and subtract. Each vanishes at the central frequency plus $1/T$ and at all other intervals separated by $1/T$.

or precisely by a Fourier harmonic; see Fig. 3. (The terminology and criterion originate in spectroscopy where the main lobe of the *sinc* function is determined by the width, L, of a physical slit playing the role of T.)

The appearance of the *sinc* function in the Fourier transform (and series) of a finite length record has some practical implications (note too, that the sampling theorem involves a sum over *sinc* functions). Suppose one has a very strong sinusoid of amplitude A, at frequency p, present in a record, $x(t)$ whose Fourier transform otherwise has a magnitude which is much less than A. If one is attempting to estimate $\hat{x}(s)$ apart from the sinusoid, one sees that the influence of A (from both positive and negative frequency contributions) will be additive and can seriously corrupt $\hat{x}(s)$ even at frequencies far from $s = p$. Such effects are known as "leakage". There are basically three ways to remove this disturbance. (1) Subtract the sinusoid from the data prior to the Fourier analysis. This is a very common procedure when dealing, e.g., with tides in sealevel records, where the frequencies are known in advance to literally astronomical precision, and where$|\hat{x}(s_p)|^2 \approx |A^2|$ may be many orders of magnitude larger than its value at other frequencies. (2) Choose a record length such that $p = n/T$; that is, make the sinusoid into a Fourier harmonic and rely on the vanishing of the *sinc* function to suppress the contribution of A at all other frequencies. This procedure is an effective one, but is limited by the extent to which finite word length computers can compute the zeros of the *sinc* and by the common problem (e.g., again for tides) that several pure frequencies are present simultaneously and not all can be rendered simultaneously as Fourier harmonics. (3) Taper the record. Here one notes that the origin of the leakage problem is that the *sinc* diminishes only as $1/s$ as one moves away from the central frequency. This slow reduction is in turn

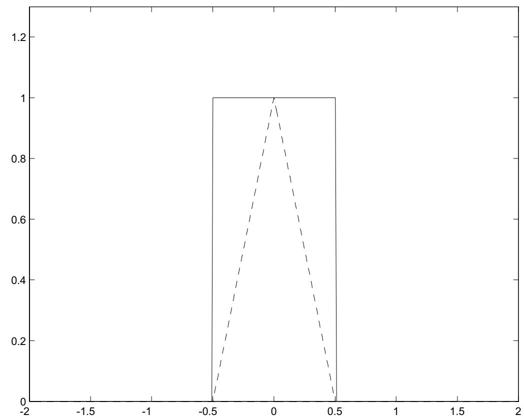

FIGURE 4. "Tophat", or $\Pi(t)$ (solid) and "triangle" or $\Lambda(t/2)$. A finite record can be regarded as the product $x(t)\Pi(t/T)$, giving rise to the *sinc* pattern response. If this finite record is tapered by multiplying it as $x(t)\Lambda(t/(2T))$, the Fourier transform decays much more rapidly away from the central frequency of any sinusoids present.

easily shown to arise because the Π function in (3.5) has finite steps in value (recall the Riemann-Lebesgue Lemma.)

Suppose we "taper" $x_\Pi(t)$, by multiplying it by the triangle function (see Bracewell, 1978, and Fig. 4),

$$\Lambda(t) = 1 - |t|, t \leq 1 \tag{3.9}$$

$$= 0, |t| > 1 \tag{3.10}$$

whose first derivative, rather than the function itself is discontinuous. The Fourier transform

$$\hat{\Lambda}(s) = \frac{\sin^2(\pi s)}{(\pi s)^2} = sinc^2(s) \tag{3.11}$$

is plotted in Fig. 2). As expected, it decays as $1/s^2$. Thus if we Fourier transform

$$x_\Lambda(t) = x(t)\Lambda(t/(T/2)) \tag{3.12}$$

the pure cosine now gives rise to

$$\hat{x}_\Lambda(s) = \frac{T}{2}\left\{\frac{\sin^2((\pi/2)T(s-p_1))}{((\pi/2)T(s-p_1))^2} + \frac{\sin^2((\pi/2)T(s+p_1))}{((\pi/2)T(s+p_1))^2}\right\} \tag{3.13}$$

and hence the leakage diminishes much more rapidly, whether or not we have succeeded in aligning the dominant cosine. A price exists however, which must be paid. Notice that the main lobe of $\mathcal{F}(\Lambda(t/(T/2)))$ has width not $2/T$, but $4/T$, that is, it is twice as wide as before, and the resolution of the analysis would be 1/2 of what it was without tapering. Thus tapering the record prior to Fourier analysis incurs a trade-off between leakage and resolution.

One might sensibly wonder if some intermediate function between the Π and Λ functions exists so that one diminishes the leakage but without incurring a resolution penalty as large as a factor of 2. The answer is "yes"; much effort has been made over the years to finding tapers $w(t)$, whose Fourier transforms $\hat{W}(s)$ have desirable properties. Such taper functions are called "windows". A common one tapers the ends by multiplying by half-cosines at either end, cosines whose periods are a parameter of the analysis. Others go under the names of Hamming, Hanning, Bartlett, etc. windows.

Later we will see that a sophisticated choice of windows leads to the elegant recent theory of multitaper spectral analysis. At the moment, we will only make the observation that the Λ taper and all other tapers, has the effect of throwing away data near the ends of the record, a process which is always best regarded as perverse: one should not have to discard good data for a good analysis procedure to work.

Although we have discussed leakage etc. for continuously sampled records, completely analogous results exist for sampled, finite, records. We leave further discussion to the references.

Exercise. Generate a pure cosine at frequency s_1, and period $T_1 = 2\pi/s_1$. Numerically compute its Fourier transform, and Fourier series coefficients, when the record length, $T =$integer $\times T_{1,}$ and when it is no longer an integer multiple of the period.