

4. Discrete Observations

4.0.1. *Sampling.* The above results show that a band-limited function can be reconstructed perfectly from an infinite set of (perfect) samples. Similarly, the Fourier transform of a time-limited function can be reconstructed perfectly from an infinite number of (perfect) samples (the Fourier Series frequencies). In observational practice, functions must be both band-limited (one cannot store an infinite number of Fourier coefficients) and time-limited (one cannot store an infinite number of samples). Before exploring what this all means, let us vary the problem slightly. Suppose we have $x(t)$ with Fourier transform $\hat{x}(s)$ and we sample $x(t)$ at uniform intervals $m\Delta t$ without paying attention, initially, as to whether it is band-limited or not. What is the relationship between the Fourier transform of the sampled function and that of $x(t)$? That is, the above development does not tell us how to compute a Fourier transform from a set of samples. One could use (3.2), interpolating before computing the Fourier integral. As it turns out, this is unnecessary.

We need some way to connect the sampled function with the underlying continuous values. The δ -function proves to be the ideal representation. Eq. (2.13) produces a single sample at time t_m . The quantity,

$$x_{III}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t), \quad (4.1)$$

vanishes except at $t = q\Delta t$ for any integer q . The value associated with $x_{III}(t)$ at that time is found by integrating it in an infinitesimal interval $-\varepsilon + q\Delta t \leq t \leq \varepsilon + q\Delta t$ and one finds immediately that $x_{III}(q\Delta t) = x(q\Delta t)$. Note that *all* measurements are integrals over some time interval, no matter how

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short (perhaps nanoseconds). Because the δ -function is infinitesimally broad in time, the briefest of measurement integrals is adequate to assign a value.¹

Let us Fourier analyze $x_{III}(t)$, and evaluate it in two separate ways:

(I) Direct sum.

$$\begin{aligned}\hat{x}_{III}(s) &= \int_{-\infty}^{\infty} x(t) \sum_{m=-\infty}^{\infty} \delta(t - m\Delta t) e^{-2\pi i s t} dt \\ &= \sum_{m=-\infty}^{\infty} x(m\Delta t) e^{-2\pi i s m\Delta t}.\end{aligned}\tag{4.2}$$

(II) By convolution.

$$\hat{x}_{III}(s) = \hat{x}(s) * \mathcal{F}\left(\sum_{m=-\infty}^{\infty} \delta(t - m\Delta t)\right).\tag{4.3}$$

What is $\mathcal{F}\left(\sum_{m=-\infty}^{\infty} \delta(t - m\Delta t)\right)$? We have, by direct integration,

$$\mathcal{F}\left(\sum_{m=-\infty}^{\infty} \delta(t - m\Delta t)\right) = \sum_{m=-\infty}^{\infty} e^{-2\pi i m s \Delta t}\tag{4.4}$$

What function is this? The right-hand-side of (4.4) is clearly a Fourier series for a function periodic with period $1/\Delta t$, in s . I assert that the periodic function is $\Delta t \delta(s)$, and the reader should confirm that computing the Fourier series representation of $\Delta t \delta(s)$ in the s -domain, with period $1/\Delta t$ is exactly (4.4). But such a periodic δ -function can also be written²

$$\Delta t \sum_{n=-\infty}^{\infty} \delta(s - n/\Delta t)\tag{4.5}$$

Thus (4.3) can be written

$$\begin{aligned}\hat{x}_{III}(s) &= \hat{x}(s) * \Delta t \sum_{n=-\infty}^{\infty} \delta(s - n/\Delta t) \\ &= \int_{-\infty}^{\infty} \hat{x}(s') \Delta t \sum_{n=-\infty}^{\infty} \delta(s - n/\Delta t - s') ds' \\ &= \Delta t \sum_{n=-\infty}^{\infty} \hat{x}\left(s - \frac{n}{\Delta t}\right)\end{aligned}\tag{4.6}$$

We now have two apparently very different representations of the Fourier transform of a sampled function. (I) Asserts two important things. The Fourier transform can be computed as the naive discretization of the complex exponentials (or cosines and sines if one prefers) times the sample values. Equally important, the result is a periodic function with period $1/\Delta t$. (Figure 5). Form (II) tells us that

¹ δ -functions are meaningful only when integrated. Lighthill (1958) is a good primer on handling them. Much of the book has been boiled down to the advice that, if in doubt about the meaning of an integral, “integrate by parts”.

²Bracewell (1978) gives a complete discussion of the behavior of these otherwise peculiar functions. Note that we are ignoring all questions of convergence, interchange of summation and integration etc. Everything can be justified by appropriate limiting processes.

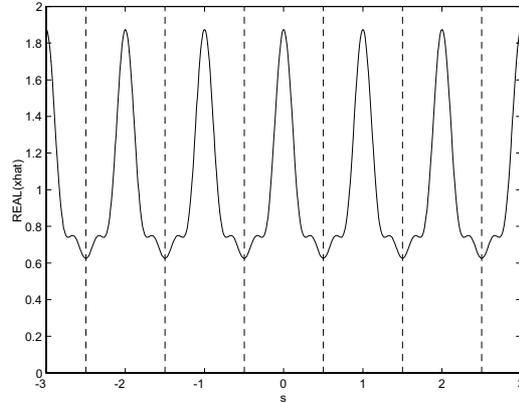


FIGURE 5. Real part of the periodic Fourier transform of a sampled function. The baseband is defined as $-1/2\Delta t \leq s \leq 1/2\Delta t$, (here $\Delta t = 1$), but any interval of width $1/\Delta t$ is equivalent. These intervals are marked with vertical lines.

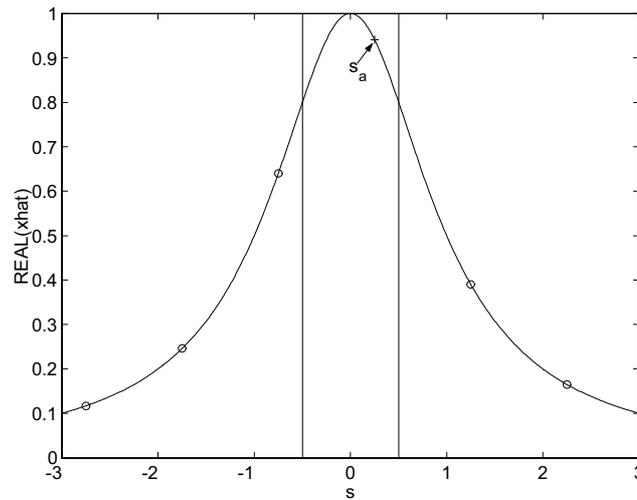


FIGURE 6. s_a is the position where all Fourier transform amplitudes from the Fourier transform values indicated by the dots (Eq. 4.6) will appear. The baseband is indicated by the vertical lines and any non-zero Fourier transform values outside this region will alias into it.

the value of the Fourier transform at a particular frequency s is not in general equal to $\hat{x}(s)$. Rather it is the sum of *all* values of $\hat{x}(s)$ separated by frequency $1/\Delta t$. (Figure 6). This second form is clearly periodic with period $1/\Delta t$, consistent with (I).

Because of the periodicity, we can confine ourselves for discussion, to one interval of width $1/\Delta t$. By convention we take it symmetric about $s = 0$, in the range $-1/(2\Delta t) \leq s \leq 1/(2\Delta t)$ which we call the

baseband. We can now address the question of when $\hat{x}_{III}(s)$ in the baseband will be equal to $\hat{x}(s)$? The answer follows immediately from (4.6): if, and only if, $\hat{x}(s)$ vanishes for $s \geq |1/2\Delta t|$. That is, the Fourier transform of a sampled function will be the Fourier transform of the original continuous function only if the original function is bandlimited and Δt is chosen to be small enough such that $\hat{x}(|s| > 1/\Delta t) = 0$. We also see that *there is no purpose in computing $\hat{x}_{III}(s)$ outside the baseband*: the function is perfectly periodic. We could use the sampling theorem to interpolate our samples before Fourier transforming. But that would produce a function which vanished outside the baseband—and we would be no wiser.

Suppose the original continuous function is

$$x(t) = A \sin(2\pi s_0 t). \quad (4.7)$$

It follows immediately from the definition of the δ -function that

$$\hat{x}(s) = \frac{i}{2} \{\delta(s + s_0) - \delta(s - s_0)\}. \quad (4.8)$$

If we choose $\Delta t < 1/2s_0$, we obtain the δ -functions in the baseband at the correct frequency. We ignore the δ -functions outside the baseband because we know them to be spurious. But suppose we choose, either knowing what we are doing, or in ignorance, $\Delta t > 1/2s_0$. Then (4.6) tells us that it will appear, spuriously, at

$$s = s_a = s_0 \pm m/\Delta t, |s_a| \leq 1/2\Delta t \quad (4.9)$$

thus determining m . The phenomenon of having a periodic function appear at an incorrect, lower frequency, because of insufficiently rapid sampling, is called “aliasing” (and is familiar through the stroboscope effect, as seen for example, in movies of turning wagon wheels).