7. Identities and Difference Equations

$Z-$transform analogues exist for all of the theorems of ordinary Fourier transforms.

Exercise. Demonstrate:

The shift theorem: $\mathcal{Z}(x_{m-q}) = z^q \hat{x}(z)$.

The differentiation theorem: $\mathcal{Z}(x_m - x_{m-1}) = (1-z)\hat{x}(z)$. Discuss the influence of a difference operation like this has on the frequency content of $\hat{x}(s)$.

The time-reversal theorem: $\mathcal{Z}(x_{-m}) = \hat{x}(1/z)$.

These and related relationships render it simple to solve many difference equations. Consider the difference equation

$$x_{m+1} - ax_m + bx_{m-1} = p_m \tag{7.1}$$

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where p_m is a known sequence and a, b are constant. To solve (7.1), take the $z-$transform of both sides, using the shift theorem:

$$\frac{1}{z}\hat{x}(z) - a\hat{x}(z) + bz\hat{x}(z) = \hat{p}(z) \tag{7.2}$$

and solving,

$$\hat{x}_p(z) = \frac{\hat{p}(z)}{(1/z - a + bz)}. \tag{7.3}$$

If $p_m = 0, m < 0$ (making p_m causal), then the solution (7.3) is both causal and stable only if the zeros of $(1/z - a + z)$ lie outside $|z| = 1$.

Exercise. Find the sequence corresponding to (7.3).

Eq. (7.3) is the particular solution to the difference equation. A second order difference equation in general requires two boundary or initial conditions. Suppose x_0, x_1 are given. Then in general we need a homogeneous solution to add to (7.3) to satisfy the two conditions. To find a homogeneous solution, take $\hat{x}_h(z) = Ac^m$ where A, c are constants. The requirement that $\hat{x}_h(z)$ be a solution to the homogeneous difference equation is evidently $c^{m+1} - ac^m + bc^{m-1} = 0$ or, $c - a + bc^{-1} = 0$, which has two roots, $c_{\pm}$. Thus the general solution is

$$x_m = \mathcal{Z}^{-1}(\hat{x}_p(z)) + A_+ c_+^m + A_- c_-^m \tag{7.4}$$

where the two constants $A_{\pm}$ are available to satisfy the two initial conditions. Notice that the roots $c_{\pm}$ determine also the character of (7.3). This is a large subject, left at this point to the references.[3]

We should note that Box, Jenkins and Reisel (1994) solve similar equations without using $z-$transforms. They instead define forward and backwards difference operators, e.g., $\mathcal{B}(x_m) = x_{m-1}, \mathcal{F}(x_m) = x_{m+1}$. It is readily shown that these operators obey the same algebraic rules as do the $z-$transform, and hence the two approaches are equivalent.

Exercise. Evaluate $(1 - \alpha\mathcal{B})^{-1} x_m$ with $|\alpha| < 1$.

[3] The procedure of finding a particular and a homogeneous solution to the difference equation is wholly analogous to the treatment of differential equations with constant coefficients.