Midterm Review

- Solow Model
- Consumption/Savings
- Ramsey problem

Solow Model

- Technology (neoclassical assumptions)

$$Y_t = F(K_t, L_t)$$

- Constant Returns to Scale:

$$y_t = \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$$

- Decreasing marginal product

$$f'(k) > 0 \qquad f''(k) < 0$$

- Inada conditions

$$\lim_{k \to 0} f'(k) = \infty \qquad \lim_{k \to \infty} f'(k) = 0$$

- Example: Cobb-Douglas

$$f(k) = k^{\alpha}$$

Capital accumulation

- Labor supply inelastic:

$$L_{t+1} = L_t\,(1+n)$$

- Save a fraction s of output:

$$K_{t+1} = K_t(1-\delta) + I_t$$

$$\text{with } I_t = sY_t$$

- Law of motion:

$$\Longrightarrow k_{t+1} \approx k_t\,(1-\delta-n) + sf(k_t)$$

$$\Longrightarrow k_{t+1} - k_t \approx sf(k_t) - (n+\delta)k_t$$

Steady State

- Find Steady State with $k_t = k_{t+1} = k^*$

$$k_{t+1} - k_t \approx sf(k_t) - (n+\delta)k_t$$

$$0 = sf(k^*) - (n+\delta)k^*$$

- With Cobb-Douglas

$$k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- Neoclassical assumptions: there is a unique positive steady state $k^* > 0$
  ($k^* = 0$ is also a SS but unstable, and not interesting)

Dynamics

- Capital converges to SS

$$k_t \to k^*(n, \delta, s)$$

- Conditional convergence: in the long-run, countries with the same (n, δ, s) should have the same k, regardless of initial conditions k_0.

Technological shocks

- Production function

$$y_t = A_t k_t^\alpha$$

- A negative shock to A_t leads to lower output per capital y_t, lower investment $i_t = sy_t$ and hence lower capital next period k_{t+1}.
- Solow residual from data

$$\Delta \log A_t = \Delta \log y_t - \underbrace{\alpha}_{=0.3} \Delta \log(k_t)$$

Competitive Markets

- The allocation of the Solow model can be achieved by a competitive equilibrium, with wage rates and rental rates for capital

$$w_t = mpl_t = \frac{\partial}{\partial L} F(K_t, L_t) = f(k_t) - f'(k_t)k_t$$

$$r_t = mpk_t = \frac{\partial}{\partial K} F(K_t, L_t) = f'(k_t)$$

- With Cobb-Douglas we get the fraction of output going to labor and capital are constant fractions

$$\frac{r_t K_t}{Y_t} = \frac{f'(k_t)k_t}{f(k_t)} = \frac{\alpha k_t^{\alpha-1} k_t}{k_t^{\alpha}} = \alpha$$

$$\frac{w_t L_t}{Y_t} = \frac{f(k_t) - f'(k_t)k_t}{f(k_t)} = 1 - \alpha$$

Empirical Evidence

- Mankiw, Romer, and Weill: Solow model works pretty well, once we account for human capital

$$Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta}$$

- Assume all countries have the same technology (α, β, δ, g) but differ in s, n, and productivity level A_0.
- The Solow model explains a big chunk of the difference in long-run output per capita across countries.
- **Business Cycles** (pset) for output per capita: not bad
- for investment (and hence capital): bad
  - investment is very procyclical: $\frac{i_t}{y_t}$ high during booms
- labor is also procyclical, not fixed.

Savings/Consumption

- Live two periods $t \in \{1, 2\}$. Chose (c_1, c_2, a_1, a_2)

$$\max u(c_1) + \beta u(c_2)$$

$$st: \quad c_1 + a_1 \leq (1 + R)a_0 + w_1$$

$$c_2 + a_2 \leq (1 + R)a_1 + w_2$$

$$a_0 \text{ given and } a_2 \geq 0$$

- Intertemporal budget constraint

$$c_1 + \frac{c_2}{1 + R} \leq \underbrace{(1 + R)a_0 + w_1 + \frac{w_2}{1 + R}}_{W}$$

- $\frac{1}{1+R}$ is the price of c_2 relative to c_1

Find optimal consumption (c_1, c_2)

$$\max u(c_1) + \beta u(c_2)$$

$$c_1 + \frac{c_2}{1+R} \leq W$$

- FOC:

$$u'(c_1) = \lambda$$
$$\beta u'(c_2) = \frac{\lambda}{1+R}$$

$$\implies u'(c_1) = \beta(1+R)u'(c_2) \quad \text{(Euler Equation)}$$

- Consumption smoothing

Use budget constraints to find (a_1, a_2)

$$a_1 = (1 + R)a_0 + w_1 - c_1$$
$$a_2 = (1 + R)a_1 + w_2 - c_2$$

- Comparative statics: $\uparrow R$
- Substitution effect: c_2 is cheaper, get more c_2 and less c_1
- Wealth effect: if $a_1 > 0$ the agent can have more c_1 and more c_2

Ramsey model

- Two differential equations for $c(t)$ and $k(t)$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+\delta)k(t) \quad \text{(feasibility)}$$

$$\frac{\dot{c}(t)}{c(t)} = \theta\,(R(t) - \rho) \quad \text{(Euler equation)}$$

$$R(t) = f'(k(t)) - \delta$$

- and two boundary conditions

$$k(0) = k_0 > 0 \quad \text{given}$$

$$\lim_{t \to \infty} k(t) = k^*$$

- where k^* is the steady state level of capital.

Steady State

- Look for a capital and consumption $(c^*, k^*) \in \mathbb{R}^2$ such that $\dot{k}(t) = \dot{c}(t) = 0$.
- We get two equations in two unknowns:

$$f(k^*) - c^* - (n + \delta)k^* = 0$$

$$R^* - \rho = 0 \quad \Longleftrightarrow \quad f'(k^*) - \delta - \rho = 0$$

- So we get for the Cobb Douglas case

$$k^* = \left(\frac{\alpha}{n + \delta} \right)^{\frac{1}{1-\alpha}}$$

Dynamics

- Draw a phse diagram by finding the combination of (c, k) such that $\dot{k}(t) = 0$ (typically a parabola looking curve) and $\dot{c}(t) = 0$ (a vertical line).
- Where they intersect we have the Steady State.
- Draw the "stable path" such that the system converges to the SS.