0.1 Ramsey problem

The production side is like in the Solow model. Output per capita

$$y_t = f(k_t)$$

simplify $n = 0$ and $g = 0$ so the law of motion for capital per capita is

$$k_{t+1} = (1 - \delta)k_t + i_t$$

$$c_t + i_t = y_t$$

$$\implies k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t$$

$$c_t = (1 - \delta)k_t + f(k_t) - k_{t+1} \tag{1}$$

with the constraints $c_t \geq 0$ and $k_{t+1} \geq 0$, and k_0 given.

In the Solow model, $c_t = (1 - s)f(k_t)$. Now instead we consider the problem of how much the planner would consume/invest. The people in the economy derive utility from consuming, and so does the planner. For a consumption stream $c = \{c_t\}_{t=0}^{T}$

$$\sum_{t=0}^{T} \beta^t u(c_t)$$

where $\beta \in (0, 1)$ is the discount factor an captures impatience and we have an infinite horizon $T = \infty$. $u(c_t)$ is the per period utility function and we assume it is

- increasing
- concave
- Inada conditions $\lim_{c \to 0} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$

Example 1. For example, the CEIS function

$$u(c) = \frac{c^{1-\frac{1}{\theta}}}{1-\frac{1}{\theta}}$$

where $\theta > 0$ is the EIS and controls how much the agent is willing to let his consumption vary across periods.

The Ramsey problem is

$$\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t u\Big(\overbrace{(1-\delta)k_t + f(k_t) - k_{t+1}}^{c_t}\Big)$$

$$st: \quad 0 \leq k_{t+1} \leq (1-\delta)k_t + f(k_t)$$

$$k_0 \text{ given}$$

Where if we have a solution $\{k_t^*\}$ then we can rebuild the sequence of consumption $\{c_t\}$ from (1) (and output $\{y_t\}$ and investment $\{i_t\}$).

Finite Horizon $T < \infty$. We can solve this problem in several ways. First imagine we have a finite horizon problem: $t = 1, ..., T$.

Then we know how to solve this problem. Ignore the non-negativity constraints (we can check them later), taking FOC (because of the concavity, FOC will be sufficient) we get

$$\beta^t u'(c_t)(-1) + \beta^{t+1} u'(c_{t+1}) \left((1-\delta) + f'(k_{t+1})\right) = 0 \quad \forall t = 1...T-1$$

$$u'(c_t) = \beta u'(c_{t+1}) \left((1-\delta) + f'(k_{t+1})\right)$$

The idea is that if we have a plan $\{k_t\}$ and we decide to reduce consumption in period t by a small ϵ and use that to invest and accumulate capital for next period $k_{t+1} + \epsilon$ then we can increase consumption next period by $(1-\delta)\epsilon + f'(k_{t+1})\epsilon$ and keep k_{t+2} and the whole subsequent plan unchanged:

$$\hat{c}_{t+1} = (1-\delta)k_{t+1} + f(k_{t+1}) + (1-\delta)\epsilon + f'(k_{t+1})\epsilon - k_{t+2}$$

The reduction in consumption at time t has a cost in utility $u'(c_t)\epsilon$ and from the increase in consumption in period $t+1$ we get $\beta u'(c_{t+1})\left((1-\delta) + f'(k_{t+1})\right)\epsilon$. It better be the case that we cannot improve by picking a small ϵ (greater or smaller than zero), and this is what the FOC condition captures: "local deviations"

So we have a second order diference equation for $\{k_t\}$:

$$u'\Big(\overbrace{(1-\delta)k_t + f(k_t) - k_{t+1}}^{c_t}\Big) = \beta u'\Big(\overbrace{(1-\delta)k_{t+1} + f(k_{t+1}) - k_{t+2}}^{c_{t+1}}\Big)\left((1-\delta) + f'(k_{t+1})\right)$$

with an initial condition k_0 given. We need a second "boundary condition". For the last period T we have $k_{T+1} = 0$, so this has a unique solution $\{k_t^*\}$. To understand this condition take the FOC for k_{T+1}:

$$\beta^T u'(c_T)(-1) \leq 0 \quad \text{and} \quad k_{T+1} \geq 0$$

because the non-negativity constraint here can be binding (this is the Kuhn Tucker condition), and with complementary slackness:

$$\beta^T u'(c_T)(-1)k_{T+1} = 0$$

So if $k_{T+1} > 0$, then $\beta^{T+1}u'(c_T)(-1) = 0$ which cannot be. Hence, $k_{T+1} = 0$. Intuitively, capital is worthless since the economy ends and we can't use it to produce consumption goods.

If this condition failed, we could at some period $t < T$ consume a little more $c_t + \epsilon$ and obtain a little bit less capital next period but not make up for it with less consumption next period (keep the same consumption for every consecutive period), so that instead of keeping k_{t+2} unchanged, it would go down a little, and so would all the consecutive k_s for $s = t + 2 ... T$. If the original consecutive capital levels k_s were strictly positive (and this will be the case for some t) this plan is still feasible, but better because we consumed more in one period and the same in all others! This is a "global" deviation.

Becuase of the concavity, the FOC - including the Kuhn Tucker inequality - are sufficient. Finally, after building the solution, check that the non-negativity constraints we ignored are actually satisfied and we are done.

Infinite Horizon $T = \infty$. With an infinite horizon $T = \infty$, we don't have a "last period" and so we never want to have no capital. The second boundary condition becomes instead a "tranversality condition":

$$\lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0$$

Proving this is beyond the scope of this course, but the intuition is similar to the finite horizon case: we don't want to accumulate capital for its own sake. The FOC conditions make sure there are no "local" deviations (consuming a little bit less today and a little bit more tomorrow), the "transversality condition" makes sure there is no "global" deviation, like simply consuming more today (without reducing consumption in the future) and having a little less capital in every consecutive period.