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**Solutions**  
**14.06 2004 - Midterm**  
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**Question 1 (Continuous Time):** (a) The household takes  $\tau, w_t, r_t$  and  $T_t$  as given and selects a consumption path to solve

$$\begin{aligned} & \max \int_0^\infty e^{-\rho t} u(c_t) dt \\ & \text{subject to} \\ \dot{k}_t &= (1 - \tau)(r_t k_t + w_t) + T_t - c_t - \delta k_t \end{aligned}$$

This is a simple optimal control problem where  $c_t$  is the control variable and  $k_t$  is the state variable. We can write the Hamiltonian as

$$H_t = e^{-\rho t} u(c_t) + \lambda_t [(1 - \tau)(r_t k_t + w_t) + T_t - c_t - \delta k_t]$$

where  $\lambda_t$  is the multiplier on the co-state equation. The necessary conditions are

$$\frac{\partial H_t}{\partial c_t} = 0 \Leftrightarrow e^{-\rho t} u'(c_t) = \lambda_t \quad (1)$$

$$\frac{\partial H_t}{\partial k_t} = -\dot{\lambda}_t \Leftrightarrow \lambda_t [(1 - \tau)r_t - \delta] = -\dot{\lambda}_t \quad (2)$$

Using (1) we have that

$$\dot{\lambda}_t = -\rho e^{-\rho t} u'(c_t) + e^{-\rho t} u''(c_t) \dot{c}_t \quad (3)$$

Putting (1) and (3) into (2) we get that

$$\begin{aligned} e^{-\rho t} u'(c_t) [(1 - \tau)r_t - \delta] &= \rho e^{-\rho t} u'(c_t) - e^{-\rho t} u''(c_t) \dot{c}_t \\ u'(c_t) [(1 - \tau)r_t - \delta - \rho] &= -u''(c_t) \dot{c}_t \\ -u''(c_t) c_t \dot{c}_t &= [(1 - \tau)r_t - \delta - \rho] \\ \frac{u'(c_t)}{u''(c_t)} \frac{\dot{c}_t}{c_t} &= [(1 - \tau)r_t - \delta - \rho] \end{aligned}$$

Using the specific form of the utility function we have that  $\frac{-u''(c_t)c_t}{u'(c_t)} = \frac{1}{\theta}$  and this gives us the households Euler equation:

$$\frac{\dot{c}_t}{c_t} = \theta [(1 - \tau)r_t - \delta - \rho]$$

Consumption growth is falling in the tax rate. A tax on capital income alters the tradeoff between saving and consumption (it taxes the former of these) and hence lowers the willingness of a household to lower consumption today in order to consume more tomorrow. Put differently, it discourages consumption growth.

(b) In equilibrium the rate of interest is simply equal to the marginal product of capital:

$$r_t = \alpha k_t^{\alpha-1}$$

This is decreasing in the level of capital. This simply comes from the fact that the Cobb-Douglas function displays diminishing returns to each individual factor.

(c) First, we know that consumption growth must obey the Euler equation. We know also that in general equilibrium the interest rate will be equal to the marginal product of capital. Accordingly we have that

$$\frac{\dot{c}_t}{c_t} = \theta [(1 - \tau)\alpha k_t^{\alpha-1} - \delta - \rho] \quad (4)$$

Next, we know that the growth of capital is governed by the resource constraint of the economy. Going through this step by step we have that the households capital stock evolves according to

$$\dot{k}_t = (1 - \tau)(r_t k_t + w_t) + T_t - c_t - \delta k_t$$

Now in general equilibrium we know that

$$\begin{aligned} k_t^\alpha &= r_t k_t + w_t \\ T_t &= \tau(r_t k_t + w_t) \end{aligned}$$

Accordingly, the resource constraint of the economy can be written as

$$\dot{k}_t = k_t^\alpha - c_t - \delta k_t \quad (5)$$

Together (4) and (5) fully characterize the dynamics of consumption and capital..

(d) The steady state level of capital is found by setting  $\dot{c}_t = 0$ . This requires that

$$(1 - \tau)\alpha k_t^{\alpha-1} = \delta + \rho$$

and hence the steady state level of capital is

$$k^* = \left( \frac{\alpha(1 - \tau)}{\delta + \rho} \right)^{\frac{1}{1-\alpha}}$$

This is decreasing in the tax rate. The tax on capital income effectively taxes saving and hence discourages saving. The result is a lower level of capital in steady state. From the perspective of the household the capital stock must be higher so that the return on capital is increased to compensate for the tax on capital income.

(e) Without drawing the phase diagram. Its clear that the  $\dot{k}_t = 0$  locus (5) is unaffected by a change in the tax rate (the proceeds of the tax are returned to the economy so the resource constraint doesn't move).

The  $\dot{c}_t = 0$  locus will move with the tax rate (in the way described above). The reduction in the tax rate will shift this locus to the right.

At the time of the change consumption will unambiguously drop (to the new saddle path). This is because now saving is more attractive and so consumers will take advantage of the higher (net) rate of return on saving. They lower consumption today to fund higher savings (with a view to achieving a higher level of consumption in the future).

Following the initial change capital rises due the increase in savings. As it rises both output and the rate of return in capital fall. This continues until the net rate of return of capital is again equal to the households discount rate. Ultimately output, consumption and capital will all end up at a higher steady state level.

**Question 1 (Discrete Time): (I Will Be Very Brief Here)**

(a) Solve this as a standard constrained optimization. The lagrangian for the households problem can be written as

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \lambda_t [k_{t+1} - (1 - \delta)k_t - (1 - \tau)(r_t k_t + w_t) - T_t + c_t]$$

where  $\lambda_t$  is the lagrange multiplier on the households budget constraint in period  $t$ . The FOC are

$$\frac{\partial L}{\partial c_t} = 0 \Leftrightarrow \beta^t u'(c_t) = \lambda_t \quad (6)$$

$$\frac{\partial L}{\partial k_{t+1}} = 0 \Leftrightarrow \lambda_t = \lambda_{t+1} [(1 - \delta) + (1 - \tau)r_t] \quad (7)$$

Putting (6) into (7) we get that

$$\begin{aligned} \beta^t u'(c_t) &= \beta^{t+1} u'(c_{t+1}) [(1 - \delta) + (1 - \tau)r_t] \\ \frac{u'(c_t)}{u'(c_{t+1})} &= \beta [(1 - \delta) + (1 - \tau)r_t] \\ \frac{c_{t+1}}{c_t} &= (\beta [(1 - \delta) + (1 - \tau)r_t])^\theta \end{aligned}$$

This is the Euler equation.

(b) See above.

(c) We use the same logic as before. The Euler equation becomes

$$\frac{c_{t+1}}{c_t} = (\beta [(1 - \delta) + (1 - \tau)\alpha k_t^{\alpha-1}])^\theta$$

and the resource constraint of the economy is

$$k_{t+1} = k_t^\alpha + (1 - \delta) k_t - c_t.$$

(d) The steady state capital level comes from setting  $\frac{c_{t+1}}{c_t} = 1$  ( $\dot{c}_t = 0$ ). This requires

$$k^* = \left[ \frac{(1 - \tau)\alpha\beta}{1 - \beta(1 - \delta)} \right]^{\frac{1}{1-\alpha}}$$

Note that this is the same as in the continuous time case when we observe that the discount factor is related to the discount rate in the following way:

$$\beta = \frac{1}{1 + \rho}.$$

(e) Same as above.