Lectures 3–4: Consumer Theory

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Consumer Theory

Consumer theory studies how rational consumer chooses what bundle of goods to consume.

Special case of general theory of choice.

Key new assumption: choice sets defined by *prices* of each of n goods, and *income* (or *wealth*).

Consumer Problem (CP)

$$\max_{x \in \mathbb{R}^n_+} u(x)$$

$$\text{s.t. } p \cdot x \leq w$$

Interpretation:

- Consumer chooses consumption vector $x = (x_1, \ldots, x_n)$
- x_k is consumption of good k
- Each unit of good k costs p_k
- Total available income is w

Lectures 3–4 devoted to studying (CP).
Lecture 5 covers some applications.

Now discuss some implicit assumptions underlying (CP).

Prices are Linear

Each unit of good k costs the same.

No quantity discounts or supply constraints.

Consumer's choice set (or **budget set**) is

$$B(p, w) = \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$$

Set is defined by single line (or hyperplane): the **budget line**

$$p \cdot x = w$$

Assume $p \geq 0$.

Goods are Divisible

$x \in \mathbb{R}^n_+$ and consumer can consume any bundle in budget set

Can model indivisibilities by assuming utility only depends on integer part of x.

Set of Goods is Finite

Debreu (1959):

> *A commodity is characterized by its physical properties, the date at which it will be available, and the location at which it will be available.*

In practice, set of goods suggests itself naturally based on context.

Marshallian Demand

The solution to the (CP) is called the **Marshallian demand** (or **Walrasian demand**).

May be multiple solutions, so formal definition is:

Definition

The **Marshallian demand correspondence** $x : \mathbb{R}^n_+ \times \mathbb{R} \rightrightarrows \mathbb{R}^n_+$ is defined by

$$\begin{aligned} x(p, w) &= \operatorname{argmax}_{x \in B(p,w)} u(x) \\ &= \left\{ z \in B(p, w) : u(z) = \max_{x \in B(p,w)} u(x) \right\}. \end{aligned}$$

Start by deriving basic properties of budget sets and Marshallian demand.

Budget Sets

Theorem

Budget sets are homogeneneous of degree 0: that is, for all $\lambda > 0$, $B(\lambda p, \lambda w) = B(p, w)$.

Proof.

$$\begin{aligned} B(\lambda p, \lambda w) &= \{x \in \mathbb{R}^n_+ | \lambda p \cdot x \leq \lambda w\} \\ &= \{x \in \mathbb{R}^n_+ | p \cdot x \leq w\} = B(p, w). \end{aligned}$$

□

Nothing changes if scale prices and income by same factor.

Theorem

If $p \gg 0$, *then* $B(p, w)$ *is compact.*

Proof.

For any p, $B(p, w)$ is closed.

If $p \gg 0$, then $B(p, w)$ is also bounded.

□

Marshallian Demand: Existence

Theorem

If u is continuous and $p \gg 0$, then (CP) has a solution. (That is, $x(p, w)$ is non-empty.)

Proof.

A continuous function on a compact set attains its maximum. □

Marshallian Demand: Homogeneity of Degree 0

Theorem
For all $\lambda > 0$, $x(\lambda p, \lambda w) = x(p, w)$.

Proof.
$B(\lambda p, \lambda w) = B(p, w)$, so (CP) with prices λp and income λw is same problem as (CP) with prices p and income w. $\square$

Marshallian Demand: Walras' Law

Theorem

If preferences are locally non-satiated, then for every (p, w) and every $x \in (p, w)$, we have $p \cdot x = w$.

Proof.

If $p \cdot x < w$, then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq B(p, w)$.
By local non-satiation, for every $\varepsilon > 0$ there exists $y \in B_\varepsilon(x)$ such that $y \succ x$.
Hence, there exists $y \in B(p, w)$ such that $y \succ x$.
But then $x \notin x(p, w)$. □

Walras' Law lets us rewrite (CP) as

$$\max_{x \in \mathbb{R}^n_+} u(x)$$

$$\text{s.t. } p \cdot x = w$$

Marshallian Demand: Differentiable Demand

Implications if demand is single-valued and differentiable:

- A proportional change in all prices and income does not affect demand:

$$\sum_{j=1}^{n} p_j \frac{\partial}{\partial p_j} x_i\left(p, w\right) + w \frac{\partial}{\partial w} x_i\left(p, w\right) = 0.$$

- A change in the price of one good does not affect total expenditure:

$$\sum_{j=1}^{n} p_j \frac{\partial}{\partial p_i} x_j\left(p, w\right) + x_i\left(p, w\right) = 0.$$

- A change in income leads to an identical change in total expenditure:

$$\sum_{i=1}^{n} p_i \frac{\partial}{\partial w} x_i\left(p, w\right) = 1.$$

The Indirect Utility Function

Can learn more about **set of solutions** to (CP) (Marshallian demand) by relating to the **value** of (CP).

Value of (CP) $=$ welfare of consumer facing prices p with income w.

The value function of (CP) is called the **indirect utility function.**

Definition

The **indirect utility function** $v : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}$ is defined by

$$v(p, w) = \max_{x \in B(p,w)} u(x).$$

Indirect Utility Function: Properties

Theorem

The indirect utility function has the following properties:

1. **Homogeneity of degree 0:** *for all* $\lambda > 0$, $v(\lambda p, \lambda w) = v(p, w)$.
2. **Continuity:** *if* u *is continuous, then* v *is continuous on* $\{(p, w) : p \gg 0, w \geq 0\}$.
3. **Monotonicty:** $v(p, w)$ *is non-increasing in* p *and non-decreasing in* w. *If* $p \gg 0$ *and preferences are locally non-satiated, then* $v(p, w)$ *is strictly increasing in* w.
4. **Quasi-convexity:** *for all* $\bar{v} \in \mathbb{R}$, *the set* $\{(p, w) : v(p, w) \leq \bar{v}\}$ *is convex.*

Indirect Utility Function: Derivatives

When indirect utility function is differentiable, its derivatives are very interesting.

Q: When is indirect utility function differentiable?
A: When u is (continuously) differentiable and Marshallian demand is unique.

For details if curious, see Milgrom and Segal (2002), "Envelope Theorems for Arbitrary Choice Sets."

Indirect Utility Function: Derivatives

Theorem

Suppose (1) u is locally non-satiated and continuously differentiable, and (2) Marshallian demand is unique in an open neighborhood of (p, w) with $p \gg 0$ and $w > 0$. Then v is differentiable at (p, w).

Furthermore, letting $x = x(p, w)$, the derivatives of v are given by:

$$\frac{\partial}{\partial w} v(p, w) = \frac{1}{p_j} \frac{\partial}{\partial x_j} u(x)$$

and

$$\frac{\partial}{\partial p_i} v(p, w) = -\frac{x_i}{p_j} \frac{\partial}{\partial x_j} u(x),$$

where j is any index such that $x_j > 0$.

Indirect Utility Function: Derivatives

$$\frac{\partial}{\partial w} v(p, w) = \frac{1}{p_j} \frac{\partial}{\partial x_j} u(x)$$

$$\frac{\partial}{\partial p_i} v(p, w) = -\frac{x_i}{p_j} \frac{\partial}{\partial x_j} u(x)$$

- Suppose consumer's income increases by \$1.
- Should spend this dollar on any good that gives biggest "bang for the buck."
- Bang for spending on good j equals $\frac{1}{p_j}\frac{\partial u}{\partial x_j}$: can buy $\frac{1}{p_j}$ units, each gives utility $\frac{\partial u}{\partial x_j}$.
- Finally, $x_j > 0$ for precisely those goods that maximize bang for buck.
- $\implies$ **marginal utility of income equals $\frac{1}{p_j}\frac{\partial u}{\partial x_j}$, for any j with $x_j > 0$.**

Indirect Utility Function: Derivatives

$$\begin{aligned} \frac{\partial}{\partial w} v(p, w) &= \frac{1}{p_j} \frac{\partial}{\partial x_j} u(x) \\ \frac{\partial}{\partial p_i} v(p, w) &= -\frac{x_i}{p_j} \frac{\partial}{\partial x_j} u(x) \end{aligned}$$

- Suppose price of good i increases by \$1.
- This effectively makes consumer $\$x_i$ poorer.
- Just saw that marginal effect of making \$1 poorer is $-\frac{1}{p_j}\frac{\partial u}{\partial x_j}$, for any j with $x_j > 0$.
- $\implies$ **marginal disutility of increase in p_i equals $-\frac{x_i}{p_j}\frac{\partial u}{\partial x_j}$, for any j with $x_j > 0$.**

Kuhn-Tucker Theorem

Theorem (Kuhn-Tucker)

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions (for some $i \in \{1, \ldots, I\}$), and consider the constrained optimization problem

$$\max_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t. } g_i(x) \geq 0 \text{ for all } i$$

If x^* is a solution to this problem (even a local solution) and a condition called **constraint qualification** *is satisfied at x^*, then there exists a vector of* **Lagrange multipliers** $\lambda = (\lambda_1, \ldots, \lambda_I)$ *such that*

$$\nabla f(x^*) + \sum_{i=1}^{I} \lambda_i \nabla g_i(x^*) = 0$$

and

$$\lambda_i \geq 0 \text{ and } \lambda_i g_i(x^*) = 0 \text{ for all } i.$$

Kuhn-Tucker Theorem: Comments

1. Any local solution to constrained optimization problem must satisfy first-order conditions of the **Lagrangian**

$$\mathcal{L}(x) = f(x) + \sum_{i=1}^{I} \lambda_i g_i(x)$$

2. Condition that $\lambda_i g_i(x^*) = 0$ for all i is called **complementary slackness**.
   - Says that multipliers on slack constraints must equal 0.
   - Consistent with interpreting λ_i as marginal value of relaxing constraint i.
3. There are different versions of constraint qualification. Simplest version: vectors $\nabla g_i(x^*)$ are linearly independent for binding constraints.
   Exercise: check that constraint qualification is always satisfied in the (CP) when $p \gg 0$, $w > 0$, and preferences are locally non-satiated.

Lagrangian for (CP)

$$\mathcal{L}(x) = u(x) + \lambda [w - p \cdot x] + \sum_{k=1}^{n} \mu_k x_k$$

$\lambda \geq 0$ is multiplier on budget constraint.
$\mu_k \geq 0$ is multiplier on the constraint $x_k \geq 0$.

FOC with respect to x_i:

$$\frac{\partial u}{\partial x_i} + \mu_i = \lambda p_i$$

Complementary slackness: $\mu_i = 0$ if $x_i > 0$. So:

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= \lambda p_i \text{ if } x_i > 0 \\ \frac{\partial u}{\partial x_i} &\leq \lambda p_i \text{ if } x_i = 0 \end{aligned}$$

Lagrangian for (CP)

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= \lambda p_i \text{ if } x_i > 0 \\ \frac{\partial u}{\partial x_i} &\leq \lambda p_i \text{ if } x_i = 0 \end{aligned}$$

Implication: marginal rate of substitution $\frac{\partial u}{\partial x_i} / \frac{\partial u}{\partial x_j}$ between any two goods consumed in positive quantity must equal the ratio of their prices p_i / p_j.

Slope of indifference curve between goods i and j must equal slope of budget line.

Intuition: equal "bang for the buck" $\frac{1}{p_i} \frac{\partial u}{\partial x_i}$ among goods consumed in positive quantity.

Back to Derivatives of v

When v is differentiable, can show:

$$\frac{\partial v}{\partial w} = \lambda \ (= \textbf{“marginal utility of income”})$$
$$\frac{\partial v}{\partial p_i} = -\lambda x_i$$

(See notes.)

Combining with $\frac{\partial u}{\partial x_j} = \lambda p_j$ if $x_j > 0$, obtain

$$\frac{\partial v}{\partial w} = \frac{1}{p_j}\frac{\partial u}{\partial x_j}$$
$$\frac{\partial v}{\partial p_i} = -\frac{x_i}{p_j}\frac{\partial u}{\partial x_j}$$

for any j with $x_j > 0$.

This proves above theorem on derivatives of v.

We've already seen the intuition.

Roy's Identity

"Increasing price of good i by \$1 is like making consumer \$$x_i$ poorer."

Corollary

Under conditions of last theorem, if $x_i(p,w) > 0$ *then*

$$x_i(p,w) = -\frac{\frac{\partial}{\partial p_i} v(p,w)}{\frac{\partial}{\partial w} v(p,w)}.$$

Key Facts about (CP), Assuming Differentiability

- Consumer's marginal utility of income equals multiplier on budget constraint: $\frac{\partial v}{\partial w} = \lambda$.
- Marginal disutility of increase in price of good i equals $-\lambda x_i$.
- Marginal utility of consumption of any good consumed in positive quantity equals λp_i.

The Expenditure Minimization Problem

In (CP), consumer chooses consumption vector to **maximize utility** subject to **maximum budget constraint**.

Also useful to study “dual” problem of choosing consumption vector to **minimize expenditure** subject to **minimum utility constraint**.

This **expenditure minimization problem (EMP)** is formally defined as:

$$\min_{x \in \mathbb{R}^n_+} p \cdot x$$

$$\text{s.t. } u(x) \geq u$$

Hicksian Demand

$$\min_{x \in \mathbb{R}^n_+} p \cdot x$$
$$\text{s.t. } u(x) \geq u$$

Hicksian demand is the set of solutions $x = h(p, u)$ to the EMP.

The **expenditure function** is the value function for the EMP:

$$e(p, u) = \min_{x \in \mathbb{R}^n_+ : u(x) \geq u} p \cdot x.$$

$e(p, u)$ is income required to attain utility u when facing prices p.

Each element of $h(p, u)$ is a consumption vector that attains utility u while minimizing expenditure given prices p.

Hicksian demand and expenditure function relate to EMP just as Marshallian demand and indirect utility function relate to CP.

Why Should we Care about the EMP?

For this course, 2 reasons:

(1) Hicksian demand useful for studying effects of price changes on "real" (Marshallian) demand.

In particular, Hicksian demand is key concept needed to decompose effect of a price change into **income and substitution effects**.

(2) Expenditure function important for welfare economics.

In particular, use expenditure function to analyze effects of price changes on consumer welfare.

Hicksian Demand: Properties

Theorem

Hicksian demand satisfies:

1. **Homogeneity of degree 0 in p:** *for all* $\lambda > 0$, $h(\lambda p, u) = h(p, u)$.
2. **No excess utility:** *if* $u(\cdot)$ *is continuous and* $p \gg 0$, *then* $u(x) = u$ *for all* $x \in h(p, u)$.
3. **Convexity/uniqueness:** *if preferences are convex, then* $h(p, u)$ *is a convex set. If preferences are strictly convex and "no excess utility" holds, then* $h(p, u)$ *contains at most one element.*

Expenditure Function: Properties

Theorem

The expenditure function satisfies:

1. **Homogeneity of degree 1 in p:** *for all* $\lambda > 0$, $e(\lambda p, u) = \lambda e(p, u)$.
2. **Continuity**: *if* $u(\cdot)$ *is continuous, then* e *is continuous in* p *and* u.
3. **Monotonicity:** $e(p, u)$ *is non-decreasing in* p *and non-decreasing in* u. *If "no excess utility" holds, then* $e(p, u)$ *is strictly increasing in* u.
4. **Concavity in p:** e *is concave in* p.

Expenditure Function: Derivatives

Shephard's Lemma: if Hicksian demand is single-valued, it coincides with the derivative of the expenditure function.

Theorem

If $u(\cdot)$ is continuous and $h(p,u)$ is single-valued, then the expenditure function is differentiable in p at (p,u), with derivatives given by

$$\frac{\partial}{\partial p_i} e(p,u) = h_i(p,u).$$

Intuition: If price of good i increases by \$1, unique optimal consumption bundle now costs $\$h_i(p,u)$ more.

Proof uses **envelope theorem**.

Envelope Theorem

Theorem (Envelope Theorem)

For $\Theta \subseteq \mathbb{R}$, let $f : X \times \Theta \to \mathbb{R}$ be a differentiable function, let $V(\theta) = \max_{x \in X} f(x, \theta)$, and let $X^*(\theta) = \{x \in X : f(x, \theta) = V(\theta)\}$.
If V is differentiable at θ then, for any $x^* \in X^*(\theta)$,

$$V'(\theta) = \frac{\partial}{\partial \theta} f(x^*, \theta).$$

Shephard's Lemma: Proof

Theorem

If $u(\cdot)$ is continuous and $h(p,u)$ is single-valued, then the expenditure function is differentiable in p at (p,u), with derivatives given by

$$\frac{\partial}{\partial p_i} e(p,u) = h_i(p,u).$$

Proof.

Recall that

$$e(p,u) = \min_{x:u(x)\geq u} p \cdot x$$

Given that e is differentiable in p, envelope theorem implies that

$$\frac{\partial}{\partial p_i} e(p,u) = \frac{\partial}{\partial p_i} p \cdot x^* = x_i^* \text{ for any } x^* \in h(p,u).$$

Comparative Statics

Comparative statics are statements about how the solution to a problem change with the parameters.

(CP): parameters are (p, w), want to know how $x(p, w)$ and $v(p, w)$ vary with p and w.

(EMP): parameters are (p, u), want to know how $h(p, u)$ and $e(p, u)$ vary with p and u.

Turns out that comparative statics of (EMP) are very simple, and help us understand comparative statics of (CP).

The Law of Demand

"Hicksian demand is always decreasing in prices."

Theorem (Law of Demand)

For every $p, p' \geq 0$, $x \in h(p, u)$, *and* $x' \in h(p', u)$, *we have*

$$(p' - p)(x' - x) \leq 0.$$

Example: if p' and p only differ in price of good i, then

$$(p'_i - p_i)(h_i(p', u) - h_i(p, u)) \leq 0.$$

Hicksian demand for a good is always decreasing in its own price.

Graphically, budget line gets steeper $\Longrightarrow$ shift along indifference curve to consume less of good 1.

The Slutsky Matrix

If Hicksian demand is differentiable, can derive an interesting result about the matrix of price-derivatives

$$D_p h(p,u) = \begin{pmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \cdots & \frac{\partial h_n(p,u)}{\partial p_1} \\ \vdots & & \vdots \\ \frac{\partial h_1(p,u)}{\partial p_n} & \cdots & \frac{\partial h_n(p,u)}{\partial p_n} \end{pmatrix}$$

This is the **Slutsky matrix.**

A $n \times n$ symmetric matrix M is **negative semi-definite** if, for all $z \in \mathbb{R}^n$, $z \cdot Mz \leq 0$.

Theorem

If $h(p,u)$ is single-valued and continuously differentiable in p at (p,u), with $p \gg 0$, then the matrix $D_p h(p,u)$ is symmetric and negative semi-definite.

Proof.

Follows from **Shephard's Lemma** and **Young's Theorem**. □

The Slutsky Matrix

What's economic content of symmetry and negative semi-definiteness of Slutsky matrix?

Negative semi-definiteness: differential version of law of demand.

Ex. if $z = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the j^{th} component, then $z \cdot D_p h(p, u) z = \frac{\partial h_i(p,u)}{\partial p_i}$, so negative semi-definiteness implies that $\frac{\partial h_i(p,u)}{\partial p_i} \leq 0$.

Symmetry: derivative of Hicksian demand for good i with respect to price of good j equals derivative of Hicksian demand for good j with respect to price of good i.

Not true for Marshallian demand, due to **income effects**.

Relation between Hicksian and Marshallian Demand

Approach to comparative statics of Marshallian demand is to relate to Hicksian demand, decompose into income and substitution effects via **Slutsky equation**.

First, relate Hicksian and Marshallian demand via simple identity:

Theorem

Suppose $u(\cdot)$ is continuous and locally non-satiated. Then:

1. *For all $p \gg 0$ and $w \geq 0$, $x(p,w) = h(p, v(p,w))$ and $e(p, v(p,w)) = w$.*
2. *For all $p \gg 0$ and $u \geq u(0)$, $h(p,u) = x(p, e(p,u))$ and $v(p, e(p,u)) = u$.*

If $v(p,w)$ is the most utility consumer can attain with income w, then consumer needs income w to attain utility $v(p,w)$.

If need income $e(p,u)$ to attain utility u, then u is most utility consumer can attain with income $e(p,u)$.

The Slutsky Equation

Theorem (Slutsky Equation)

Suppose $u(\cdot)$ is continuous and locally non-satiated. Let $p \gg 0$ and $w = e(p,u)$. If $x(p,w)$ and $h(p,u)$ are single-valued and differentiable, then, for all i, j,

$$\underbrace{\frac{\partial x_i(p,w)}{\partial p_j}}_{\text{total effect}} = \underbrace{\frac{\partial h_i(p,u)}{\partial p_j}}_{\text{substitution effect}} \underbrace{- \frac{\partial x_i(p,w)}{\partial w} x_j(p,w)}_{\text{income effect}}.$$

Intuition: If p_j increases, two effects on demand for good i:

- **Substitution effect:** $\frac{\partial h_i(p,u)}{\partial p_j}$
  - Movement along original indifference curve.
  - Response to change in prices, holding utility fixed.
- **Income effect:** $-\frac{\partial x_i(p,w)}{\partial w} x_j(p,w)$
  - Movement from one indifference curve to another.
  - Response to change in income, holding prices fixed.

Terminology for Consumer Theory Comparative Statics

Definition

Good i is a **normal good** if $x_i\left(p, w\right)$ is increasing in w.
It is an **inferior good** if $x_i\left(p, w\right)$ is decreasing in w.

Definition

Good i is a **regular good** if $x_i\left(p, w\right)$ is decreasing in p_i.
It is a **Giffen good** if $x_i\left(p, w\right)$ is increasing in p_i.

Definition

Good i is a **substitute** for good j if $h_i\left(p, u\right)$ is increasing in p_j.
It is a **complement** if $h_i\left(p, u\right)$ is decreasing in p_j.

Definition

Good i is a **gross substitute** for good j if $x_i\left(p, u\right)$ is increasing in p_j.
It is a **gross complement** if $x_i\left(p, u\right)$ is decreasing in p_j.

Comparative Statics: Remarks

- Both the substitution effect and the income effect can have either sign.
  - Substitution effect is positive for substitutes and negative for complements.
  - Income effect is negative for normal goods and positive for inferior goods.
- By symmetry of Slutsky matrix, i is a substitute for $j \Leftrightarrow j$ is a substitute for i.
- **Not** true that i is a gross substitute for $j \Leftrightarrow j$ is a gross substitute for i.
  - Income effects are not symmetric.

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