Lecture 9: Attitudes toward Risk

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Money Lotteries

Today: special case of choice under uncertainty where outcomes are measured in dollars.

Set of consequences C is subset of $\mathbb{R}$.

A lottery is a cumulative distribution function F on $\mathbb{R}$.

Assume preferences have expected utility representation:

$$U(F) = E_F[u(x)] = \int u(x)\, dF(x)$$

Assume u increasing, differentiable.

Question: how do properties of von Neumann-Morgenstern utility function u relate to decision-maker's attitude toward risk?

Expected Value vs. Expected Utility

Expected **value** of lottery F is

$$E_F\left[x\right] = \int x dF\left(x\right)$$

Expected **utility** of lottery F is

$$E_F\left[u\left(x\right)\right] = \int u\left(x\right) dF\left(x\right)$$

Can learn about consumer's risk attitude by comparing $E_F\left[u\left(x\right)\right]$ and $u\left(E_F\left[x\right]\right)$.

Risk Attitude: Definitions

Definition

A decision-maker is **risk-averse** if she always prefers the sure wealth level $E_F[x]$ to the lottery F: that is,

$$\int u(x)\,dF(x) \leq u\left(\int x dF(x)\right) \text{ for all } F.$$

A decision-maker is **strictly risk-averse** if the inequality is strict for all non-degenerate lotteries F.

A decision-maker is **risk-neutral** if she is always indifferent:

$$\int u(x)\,dF(x) = u\left(\int x dF(x)\right) \text{ for all } F.$$

A decision-maker is **risk-loving** if she always prefers the lottery:

$$\int u(x)\,dF(x) \geq u\left(\int x dF(x)\right) \text{ for all } F.$$

Risk Aversion and Concavity

Statement that $\int u(x)\, dF(x) \leq u\left(\int x dF(x)\right)$ for all F is called **Jensen's inequality**.

Fact: Jensen's inequality holds iff u is concave.

This implies:

Theorem

A decision-maker is (strictly) risk-averse if and only if u is (strictly) concave.
A decision-maker is risk-neutral if and only if u is linear.
A decision-maker is (strictly) risk-loving if and only if u is (strictly) convex.

Certainty Equivalents

Can also define risk-aversion using **certainty equivalents**.

Definition

The **certainty equivalent** of a lottery F is the sure wealth level that yields the same expected utility as F: that is,

$$CE(F, u) = u^{-1}\left(\int u(x)\, dF(x)\right).$$

Theorem

A decision-maker is risk-averse iff $CE(F, u) \leq E_F(x)$ *for all* F.
A decision-maker is risk-neutral iff $CE(F, u) = E_F(x)$ *for all* F.
A decision-maker is risk-loving iff $CE(F, u) \geq E_F(x)$ *for all* F.

Quantifying Risk Attitude

We know what it means for a consumer to be risk-averse.
What does it mean for one consumer to be **more** risk-averse than another?

Two possibilities:

1. u is more risk-averse than v if, for every F, $CE(F,u) \leq CE(F,v)$.
2. u is more risk-averse than v if u is "more concave" than v, in that $u = g \circ v$ for some increasing, concave g.

One more, based on local curvature of utility function:
u is more-risk averse than v if, for every x,

$$-\frac{u''(x)}{u'(x)} \geq -\frac{v''(x)}{v'(x)}$$

$A(x,u) = -\frac{u''(x)}{u'(x)}$ is called the **Arrow-Pratt coefficient of absolute risk-aversion**.

An Equivalence

Theorem

The following are equivalent:

1. *For every* F, $CE(F, u) \leq CE(F, v)$.
2. *There exists an increasing, concave function* g *such that* $u = g \circ v$.
3. *For every* x, $A(x, u) \geq A(x, v)$.

Risk Attitude and Wealth Levels

How does risk attitude vary with wealth?

Natural to assume that a richer individual is **more willing to bear risk**: whenever a poorer individual is willing to accept a risky gamble, so is a richer individual.

Captured by **decreasing absolute risk-aversion**:

Definition

A von Neumann-Morenstern utility function u exhibits **decreasing (constant, increasing) absolute risk-aversion** if $A(x, u)$ is decreasing (constant, increasing) in x.

Risk Attitude and Wealth Levels

Theorem

Suppose u exhibits decreasing absolute risk-aversion.
If the decision-maker accepts some gamble at a lower welath level, she also accepts it at any higher wealth level: that is, for any lottery $F(x)$, if

$$E_F\left[u\left(w+x\right)\right] \geq u\left(w\right),$$

then, for any $w' > w$,

$$E_F\left[u\left(w'+x\right)\right] \geq u\left(w'\right).$$

Multiplicative Gambles

What about gambles that **multiply** wealth, like choosing how risky a stock portfolio to hold?

Are richer individuals also more willing to bear multiplicative risk?

Depends on increasing/decreasing **relative risk-aversion:**

$$R(x, u) = -\frac{u''(x)}{u'(x)}x.$$

Theorem

Suppose u exhibits decreasing relative risk-aversion.

If the decision-maker accepts some multiplicative gamble at a lower wealth level, she also accepts it at any higher wealth level: that is, for any lottery $F(t)$, if

$$E_F[u(tw)] \geq u(w),$$

then, for any $w' > w$,

$$E_F\left[u\left(tw'\right)\right] \geq u\left(w'\right).$$

Relative Risk-Aversion vs. Absolute Risk-Aversion

$$R(x) = xA(x)$$

decreasing relative risk-aversion $\implies$ decreasing absolute risk-aversion

increasing absolute risk-aversion $\implies$ increasing relative risk-aversion

Ex. decreasing relative risk-aversion $\implies$ more willing to gamble 1% of wealth as get richer.
So certainly more willing to gamble a fixed amount of money.

Application: Insurance

Risk-averse agent with wealth w, faces probability p of incurring monetary loss L.

Can insure against the loss by buying a policy that pays out a if the loss occurs.

Policy that pays out a costs qa.

How much insurance should she buy?

Agent's Problem

$$\max_a pu\left(w - qa - L + a\right) + \left(1 - p\right) u\left(w - qa\right)$$

u concave $\implies$ concave problem, so FOC is necessary and sufficient.

FOC:

$$p\left(1 - q\right) u'\left(w - qa - L + a\right) = \left(1 - p\right) qu'\left(w - qa\right)$$

Equate marginal benefit of extra dollar in each state.

Actuarily Fair Prices

Insurance is **actuarily fair** if expected payout qa equals cost of insurance pa: that is, $p = q$.

With acturarily fair insurance, FOC becomes

$$u'(w - qa - L + a) = u'(w - qa)$$

Solution: $a = L$

A risk-averse consumer facing actuarily fair prices will **always** fully insure.

Actuarily Unfair Prices

What if insurance company makes a profit, so $q > p$?

Rearrange FOC as

$$\frac{u'\left(w - qa - L + a\right)}{u'\left(w - qa\right)} = \frac{\left(1 - p\right) q}{p\left(1 - q\right)} > 1$$

Solution: $a < L$

A risk-averse consumer facing actuarily unfair prices will **never** fully insure.

Intuition: u approximately linear for small risks, so not worth giving up expected value to insure away last little bit of variance.

Comparative Statics

$$\max_a pu\left(w - qa - L + a\right) + \left(1 - p\right) u\left(w - qa\right)$$

Bigger loss $\implies$ buy more insurance (a^* increasing in L)
Follows from Topkis' theorem.

If agent has decreasing absolute risk-aversion, then she buys less insurance as she gets richer.
See notes for proof.

Application: Portfolio Choice

Risk-averse agent with wealth w has to invest in a safe asset and a risky asset.

Safe asset pays certain return r.

Risky asset pays random return z, with cdf F.

Agent's problem

$$\max_{a \in [0,w]} \int u\left(az + (w-a)\,r\right) dF\left(z\right)$$

First-order condition

$$\int \left(z - r\right) u'\left(az + (w-a)\,r\right) dF\left(z\right) = 0$$

Risk-Neutral Benchmark

Suppose $u'(x) = \alpha x$ for some $\alpha > 0$.

Then

$$U(a) = \int \alpha \left(az + (w - a) r \right) dF(z),$$

so

$$U'(a) = \alpha \left(E[z] - r \right).$$

Solution: set $a = w$ if $E[z] > r$, set $a = 0$ if $E[z] < r$.

Risk-neutral investor puts **all** wealth in the asset with the highest rate of return.

r>E[z] Benchmark

$$U'(0) = \int (z - r)\, u'(w)\, dF = (E[z] - r)\, u'(w)$$

If safe asset has higher rate of return, then even risk-averse investor puts **all** wealth in the safe asset.

More Interesting Case

What if agent is risk-averse, but risky asset has higher expected return?

$$U'(0) = (E[z] - r)\, u'(w) > 0$$

If risky asset has higher rate of return, then risk-averse investor always puts **some** wealth in the risky asset.

Comparative Statics

Does a less risk-averse agent always invest more in the risky asset?

Sufficient condition for agent v to invest more than agent u:

$$\int (z-r)\, u'\left(az+(w-a)\,r\right) dF = 0$$

$$\implies \int (z-r)\, v'\left(az+(w-a)\,r\right) dF \geq 0$$

u more risk-averse $\implies$ $v = h \circ u$ for some increasing, convex h.

Inequality equals

$$\int (z-r)\, h'\left(u\left(az+(w-a)\,r\right)\right) u'\left(az+(w-a)\,r\right) dF \geq 0$$

$h'(\cdot)$ positive and increasing in z
$\implies$ multiplying by $h'(\cdot)$ puts more weight on positive $(z>r)$ terms, less weight on negative terms.

A less risk-averse agent **always** invests more in the risky asset.

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