Lecture 10: Comparing Risky Prospects

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Risky Prospects

Last class: studied decision-maker's subjective attitude toward risk.

This class: study objective properties of risky prospects (lotteries, gambles) themselves, relate to individual decision-making.

Topics:

- First-Order Stochastic Dominance
- Second-Order Stochastic Dominance
- (Optional) Some recent research extending these concepts

First-Order Stochastic Dominance

When is one lottery unambiguously better than another?

Natural definition: F dominates G if, for every amount of money x, F is more likely to yield at least x dollars than G is.

Definition

For any lotteries F and G over $\mathbb{R}$, F **first-order stochastically dominates (FOSD)** G if

$$F(x) \leq G(x) \text{ for all } x.$$

FOSD and Choice

Main theorem relating FOSD to decision-making:

Theorem

F FOSD G iff **every** *decision-maker with a non-decreasing utility function prefers F to G.*
That is, the following are equivalent:

1. $F(x) \leq G(x)$ *for all* x.
2. $\int u(x)\, dF \geq \int u(x)\, dG$ *for every non-decreasing function* $u : \mathbb{R} \to \mathbb{R}$.

Preferred by Everyone => FOSD

If F does **not** FOSD G, then there's some amount of money x^* such that G is more likely to give at least x^* than F is.

Consider a consumer who only cares about getting at least x^* dollars.

She will prefer G.

FOSD => Preferred by Everyone

Main idea: F FOSD $G \implies F$ gives more money "realization-by-realization."

Suppose draw x according to G, but then instead give decision-maker

$$y(x) = F^{-1}(G(x))$$

Then:

1. $y(x) \geq x$ for all x, and
2. y is distributed according to F.

$\implies$ paying decision-maker according to F just like first paying according to G, then sometimes giving more money.

Any decision-maker who likes money likes this.

Second-Order Stochastic Dominance

Q: When is one lottery better than another for any decision-maker?
A: First-Order Stochastic Dominance.

Q: When is one lottery better than another for any **risk-averse** decision-maker?
A: Second-Order Stochastic Dominance.

Definition

F **second-order stochastically dominates (SOSD)** G iff every decision-maker with a non-decreasing and concave utility function prefers F to G: that is,

$$\int u(x)\, dF \geq \int u(x)\, dG$$

for every non-decreasing and concave function $u : \mathbb{R} \to \mathbb{R}$.

SOSD is a weaker property than FOSD.

SOSD for Distributions with Same Mean

If F and G have same mean, when will any risk-averse decision-maker prefer F?

When is F "unambiguously less risky" than G?

Mean-Preserving Spreads

G is a **mean-preserving spread** of F if G can be obtained by first drawing a realization from F and then adding noise.

Definition

G is a **mean-preserving spread** of F iff there exist random variables x, y, and ε such that

$$y = x + \varepsilon,$$

x is distributed according to F, y is distributed according to G, and $E\left[\varepsilon|x\right] = 0$ for all x.

Formulation in terms of cdfs:

$$\int_{-\infty}^{x} G\left(y\right) dy \geq \int_{-\infty}^{x} F\left(y\right) dy \text{ for all } x.$$

Characterization of SOSD for CDFs with Same Mean

Theorem

Assume that $\int xdF = \int xdG$. Then the following are equivalent:

1. *F SOSD G.*
2. *G is a mean-preserving spread of F.*
3. $\int_{-\infty}^{x} G(y)\, dy \geq \int_{-\infty}^{x} F(y)\, dy$ *for all x.*

General Characterization of SOSD

Theorem

The following are equivalent:

1. F *SOSD* G.
2. $\int_{-\infty}^{x} G(y)\, dy \geq \int_{-\infty}^{x} F(y)\, dy$ *for all* x.
3. *There exist random variables* x, y, z, *and* ε *such that*

$$y = x + z + \varepsilon,$$

   x *is distributed according to* F, y *is distributed according to* G, z *is always non-positive, and* $E[\varepsilon|x] = 0$ *for all* x.
4. *There exists a cdf* H *such that* F *FOSD* H *and* G *is a mean-preserving spread of* H.

Complete Dominance Orderings [Optional]

FOSD and SOSD are **partial** orders on lotteries:
"most distributions" are not ranked by FOSD or SOSD.

To some extent, nothing to be done:
If F doesn't FOSD G, some decision-maker prefers G.
If F doesn't SOSD G, some risk-averse decision-maker prefers G.

However, recent series of papers points out that if view F and G as lotteries over monetary gains and losses rather than final wealth levels, and only require that no decision-maker prefers G to F **for all wealth levels**, do get a complete order on lotteries (and index of lottery's "riskiness").

Acceptance Dominance

Consider decision-maker with wealth w, has to accept or reject a gamble F over gains/losses x.

Accept iff

$$E_F\left[u\left(w+x\right)\right]\geq u\left(w\right).$$

Definition

F **acceptance dominates** G if, whenever F is rejected by decision-maker with concave utility function u and wealth w, so is G.

That is, for all u concave and $w>0$,

$$\begin{aligned} E_F\left[u\left(w+x\right)\right] &\leq u\left(w\right) \\ &\Longrightarrow \\ E_G\left[u\left(w+x\right)\right] &\leq u\left(w\right). \end{aligned}$$

Acceptance Dominance and FOSD/SOSD

F SOSD G
$\implies E_F[u(w+x)] \geq E_G[u(w+x)]$ for all concave u and wealth w
$\implies F$ acceptance dominates G.

If $E_F[x] > 0$ but x can take on both positive and negative values, can show that F acceptance dominates lottery that doubles all gains and losses.

Acceptance dominance refines SOSD.
But still very incomplete.

Turns out can get complete order from something like: acceptance dominance at all wealth levels, or for all concave utility functions.

Wealth Uniform Dominance

Definition

F **wealth-uniformly dominates** G if, whenever F is rejected by decision-maker with concave utility function u at **every** wealth level w, so is G.
That is, for all $u \in \mathcal{U}^*$,

$$
\begin{aligned}
E_F\left[u\left(w+x\right)\right] &\leq u\left(w\right) \text{ for all } w>0 \\
&\Longrightarrow \\
E_G\left[u\left(w+x\right)\right] &\leq u\left(w\right) \text{ for all } w>0.
\end{aligned}
$$

Utility Uniform Dominance

Definition

F **utility-uniformly dominates** G if, whenever F is rejected at wealth level w by a decision-maker with **any** utility function $u \in \mathcal{U}^*$, so is G.

That is, for all $w > 0$,

$$\begin{aligned} E_F\left[u\left(w + x\right)\right] &\leq u\left(w\right) \text{ for all } u \in \mathcal{U}^* \\ &\Longrightarrow \\ E_G\left[u\left(w + x\right)\right] &\leq u\left(w\right) \text{ for all } u \in \mathcal{U}^*. \end{aligned}$$

Uniform Dominance: Results

Hart (2011):

- Wealth-uniform dominance and utility-uniform dominance are complete orders.
- Comparison of two lotteries in these orders boils down to comparison of simple measures of the "riskiness" of the lotteries.
- Measure for wealth-uniform dominance: critical level of risk-aversion above which decision maker with constant absolute risk-aversion rejects the lottery.
- Measure for utility–uniform dominance: critical level of wealth below which decision-maker with log utility rejects the lottery.