Handout on taxing externalities with uniform taxes

Let A, mnemonic for atmosphere, be the only route for externalities.

$$A = \sum_i a^i x_1^i \tag{1}$$

Note A is the same for all consumers. Note linearity is not important. This could be done with a vector of different atmospheres.

I. Quasi-linear preferences, 2 goods, additive externality

$$\begin{aligned} u^h &= x_0^h + f^h\left(x_1^h\right) + g^h\left(A\right); \quad f'^h > 0, \ \ f''^h < 0 \\ x_0^h &+ q x_1^h = I^h \end{aligned} \tag{2}$$

Consumers ignore atmospheric feedback.
Interior solution

$$f'^h\left(x_1^h\right) = q \ \rightarrow \ x_1^h\left(q\right), \ \ x_1'^h \le 0 \tag{3}$$

Social Welfare assuming linear technology with producer price p for good one.

$$\begin{aligned} \sum_h u^h &= \sum_h x_0^h + \sum_h f^h\left[x_1^h\left(q\right)\right] + \sum_h g^h\left[A\right] \\ &= \sum_h x_0^h + \sum_h f^h\left[x_1^h\left(q\right)\right] + \sum_h g^h\left[\sum_i a^i x_1^i\left(q\right)\right] \\ &= \sum_h e_0^h - p\sum_h x_1^h\left(q\right) + \sum_h f^h\left[x_1^h\left(q\right)\right] + \sum_h g^h\left[\sum_i a^i x^i\left(q\right)\right] \end{aligned} \tag{4}$$

First order condition from differentiating (4) with respect to q:

$$\begin{aligned} 0 &= -p\sum_h x_1'^h + \sum_h f'^h x_1'^h + \left(\sum_h g'^h\right)\left(\sum_i a^i x_1'^i\right) \\ &= -p\sum_h x_1'^h + q\sum_h x_1'^h + \left(\sum_h g'^h\right)\left(\sum_i a^i x_1'^i\right) \end{aligned} \tag{5}$$

$$q - p = t = -\left(\sum_h g'^h\right)\left(\sum_i a^i x_1'^i\right) / \left(\sum_i x_1'^i\right) = -\left(\sum_h g'^h\right)\frac{\sum_i a^i x_1'^i}{\sum_i x_1'^i} \tag{6}$$

II. Quasi-linear preferences, 2 goods, externality not necessarily additive

$$u^h = x_0^h + f^h\left(x_1^h, A\right) \tag{7}$$

$$\frac{\partial f^h\left(x_1^h, A\right)}{\partial x_1^h} = q \quad \rightarrow \quad x_1^h\left(q, A\right) \tag{8}$$

Solve (8) and (1) simultaneously. Note $\partial x_1^h / \partial q\,|_A$ different from $\frac{d}{dq} x_1^h\left(q, A(q)\right)$.

III. Quasi-linear preferences, 3 goods, additive externality

$$u^h = x_0^h + f^h\left(x_1^h, x_2^h\right) + g^h\left(\sum_i a^i x_1^i\right) \tag{9}$$

$$\begin{aligned} &f_1^h\left(x_1^h, x_2^h\right) = q_1 \\ &f_2^h\left(x_1^h, x_2^h\right) = q_2 \\ &x_i^h\left(q_1, q_2\right) \end{aligned} \tag{10}$$

Social welfare function

$$\sum_h e^h - p_1 \sum_h x_1^h\left(q_1, q_2\right) - p_2 \sum_h x_2^h\left(q_1, q_2\right) + \sum_h f^h\left[x_1^h\left(q_1, q_2\right), x_2^h\left(q_1, q_2\right)\right] + \sum_h g^h\left(A\right) \tag{11}$$

First order conditions with respect to q_1 and q_2

$$\begin{aligned} 0 &= t_1 \sum_h \frac{\partial x_1^h}{\partial q_1} + t_2 \sum_h \frac{\partial x_2^h}{\partial q_1} + \left(\sum_h g'^h\right)\left(\sum_i a^i \frac{\partial x_1^i}{\partial q_1}\right) \\ 0 &= t_1 \sum_h \frac{\partial x_1^h}{\partial q_2} + t_2 \sum_h \frac{\partial x_2^h}{\partial q_2} + \left(\sum_h g'^h\right)\left(\sum_i a^i \frac{\partial x_1^i}{\partial q_2}\right) \end{aligned} \tag{12}$$

If, at the optimum we have $\sum_i a^i \frac{\partial x_1^i}{\partial q_1} \neq \sum_i a^i \frac{\partial x_1^i}{\partial q_2}$, then at the optimum $t_2 \neq 0$.

IV. General preferences, additive externality, lump sum redistribution, two goods

$$\begin{aligned} &\text{Max} \ \sum_h \left\{ v^h\left(q, I^h\right) + g^h\left[\sum_i a^i x_1^i\left(q, I^i\right)\right]\right\} \\ &\text{s.t.} \quad p \cdot \sum_h x^h\left(q, I^h\right) = B \end{aligned} \tag{13}$$

Normalizations: $p_0 = q_0 = 1; \ p_1 = p; \ q_1 = q$

Alternative writing of the resource constraint:

$$\sum_h \left\{ x_0{}^h\left(q, I^h\right) + p x_1{}^h\left(q, I^h\right)\right\} = \sum_h e_0{}^h \tag{14}$$

FOC:

$$\sum_h \frac{\partial v^h}{\partial q_1} - \lambda \sum_h \left\{ \frac{\partial x_0{}^h}{\partial q_1} + p \frac{\partial x_1{}^h}{\partial q_1} \right\} + \left(\sum_h g'^h \right)\left(\sum_i a^i \frac{\partial x_1^i}{\partial q_1} \right) = 0 \tag{15}$$

Using Roy's identity and moving the last term:

$$-\sum_h x_1{}^h \frac{\partial v^h}{\partial I} - \lambda \sum_h \left(\frac{\partial x_0{}^h}{\partial q_1} + p \frac{\partial x_1{}^h}{\partial q_1} \right) = -\left(\sum_h g'^h \right)\left(\sum_i a^i \frac{\partial x_1^i}{\partial q_1} \right) \tag{16}$$

From the individual budget constraints we have:

$$\begin{aligned} &x_0^h(q, I) + q x_1^h(q, I) = I \\ &x_1^h(q, I) + \frac{\partial x_0^h}{\partial q} + q \frac{\partial x_1^h}{\partial q} = 0 \\ &\frac{\partial x_0^h}{\partial I} + q \frac{\partial x_1^h}{\partial I} = 1 \end{aligned} \tag{17}$$

With $q = p + t$, we have:

$$\begin{aligned} &\frac{\partial x_0^h}{\partial q} + p \frac{\partial x_1^h}{\partial q} = -t \frac{\partial x_1^h}{\partial q} - x_1^h(q, I) \\ &\frac{\partial x_0^h}{\partial I} + p \frac{\partial x_1^h}{\partial I} = 1 - t \frac{\partial x_1^h}{\partial I} \end{aligned} \tag{18}$$

Substituting in (16) we have:

$$-\sum_h x_1^h \frac{\partial v^h}{\partial I} + \lambda\left(\sum_h x_1^h + t\sum_h \frac{\partial x_1^h}{\partial q_1}\right) = -\left(\sum_h g'^h\right)\left(\sum_i a^i \frac{\partial x_1^i}{\partial q_1}\right) \qquad (19)$$

We have FOC for each I^h:

$$\frac{\partial v^h}{\partial I} - \lambda\left(\frac{\partial x_0^h}{\partial I} + p\frac{\partial x_1^h}{\partial I}\right) + \left(\sum_h g'^h\right) a^h \frac{\partial x_1^h}{\partial I} = 0 \qquad (20)$$

Substituting from the budget constraints:

$$\frac{\partial v^h}{\partial I} = \lambda - \lambda t \frac{\partial x_1^h}{\partial I} - \left(\sum_h g'^h\right) a^h \frac{\partial x_1^h}{\partial I} \qquad (21)$$

The Slutsky equation

$$s_1^h = \frac{\partial x_1^h}{\partial q_1} + x_1^h \frac{\partial x_1^h}{\partial I} \qquad (22)$$

Substituting from the Slutsky equation in (19):

$$-\sum_h x_1^h \frac{\partial v^h}{\partial I} + \lambda\left(\sum_h x_1^h + t\sum_h\left(s_1^h - x_1^h \frac{\partial x_1^h}{\partial I}\right)\right) = -\left(\sum_h g'^h\right)\left(\sum_i a^i\left(s_1^i - x_1^i \frac{\partial x_1^i}{\partial I}\right)\right) \qquad (23)$$

Rearranging terms:

$$-\sum_h x_1^h \frac{\partial v^h}{\partial I} + \lambda\left(\sum_h x_1^h + t\sum_h s_1^h\right) = -\left(\sum_h g'^h\right)\left(\sum_i a^i s_1^i\right) + \left(\sum_h g'^h\right)\left(\sum_i a^i x_1^i \frac{\partial x_1^i}{\partial I}\right) + \lambda t \sum_h\left(x_1^h \frac{\partial x_1^h}{\partial I}\right)$$

(24)

Substituting from (21):

$$-\sum_h x_1^h \frac{\partial v^h}{\partial I} + \lambda\left(\sum_h x_1^h + t\sum_h s_1^h\right) = -\left(\sum_h g'^h\right)\left(\sum_i a^i s_1^i\right) - \sum_h\left(x_1^h\left(\frac{\partial v^h}{\partial I} - \lambda\right)\right) \qquad (25)$$

Simplifying, we have:

$$t = -\left(\frac{\sum_h g'^h}{\lambda}\right)\left(\frac{\sum_h a^h s_1^h}{\sum_h s_1^h}\right) \tag{26}$$

Adding (21) over h:

$$\sum_h \frac{\partial v^h}{\partial I} - \lambda H = -\lambda t \sum_h \frac{\partial x_1^h}{\partial I} - \left(\sum_h g'^h\right)\left(\sum_h a^h \frac{\partial x_1^h}{\partial I}\right) \tag{27}$$

Substituting for λt:

$$\lambda H - \sum_h \frac{\partial v^h}{\partial I} = -\left(\sum_h g'^h\right)\left(\sum_h a^h s_1^h\right)\left(\sum_h \frac{\partial x_1^h}{\partial I}\right) / \sum_h \left(s_1^h\right) + \left(\sum_h g'^h\right)\left(\sum_h a^h \frac{\partial x_1^h}{\partial I}\right) \tag{28}$$

or

$$\left(\lambda H - \sum_h \frac{\partial v^h}{\partial I}\right) = \left(\sum_h \frac{\partial x_1^h}{\partial I}\right)\left(\sum_h g'^h\right)\left\{\sum_h \left(a^h \frac{\partial x_1^h}{\partial I}\right) / \left(\sum_h \frac{\partial x_1^h}{\partial I}\right) - \left(\sum_h a^h s_1^h\right) / \left(\sum_h s_1^h\right)\right\} \tag{29}$$

or

$$\lambda H = \sum_h \frac{\partial v^h}{\partial I} + \sum_h \left(\frac{\partial x_1^h}{\partial I}\right)\left(\sum_h g'^h\right)\left\{\frac{\sum_h a^h \frac{\partial x_1^h}{\partial I}}{\sum_h \frac{\partial x_1^h}{\partial I}} - \frac{\sum_h a^h s_1^h}{\sum_h s_1^h}\right\} \tag{30}$$