Handout 2 on inefficiency with incomplete markets

I. Change in production.

Trading in each state of nature – no trade across states – production decision before state is known

Consumer choice for type A

$$\begin{aligned} &\max \sum_s \pi_s u^A\left(x_{0s}^A, x_{1s}^A\right) \\ &\text{s.t. } \mathrm{x}_{0s}^{\mathrm{A}} + p_s x_{1s}^A = e_0^A, \quad s = 1,2 \end{aligned} \tag{1}$$

$$\pi_s u_0^A\left(x_{0s}^A, x_{1s}^A\right) = \lambda_s^A; \quad \pi_s u_1^A\left(x_{0s}^A, x_{1s}^A\right) = \lambda_s^A p_s \tag{2}$$

By Roy's identity, we have

$$\frac{d\sum_s \pi_s u^A\left(x_{0s}^A, x_{1s}^A\right)}{dp_s} = -\pi_s u_0^A\left(x_{0s}^A, x_{1s}^A\right) x_{1s}^A \tag{3}$$

Consumer/producer choice for type B

$$\begin{aligned} &\max \sum_s \pi_s u^B\left(x_{0s}^B, x_{1s}^B\right) \\ &\text{s.t. } \mathrm{x}_{0s}^{\mathrm{B}} + p_s x_{1s}^B = e_0^B + p_s e_{1s}^B, \quad s = 1,2 \\ &F\left(e_{11}^B, e_{12}^B\right) = 0 \end{aligned} \tag{4}$$

$$\frac{F_1}{F_2} = \frac{\lambda_1^B p_1}{\lambda_2^B p_2} = \frac{\pi_1 u_0^B\left(x_{01}^B, x_{11}^B\right) p_1}{\pi_2 u_0^B\left(x_{02}^B, x_{12}^B\right) p_2} = \frac{\pi_1 u_1^B\left(x_{01}^B, x_{11}^B\right)}{\pi_2 u_1^B\left(x_{02}^B, x_{12}^B\right)} = \frac{\pi_1 u_1^B(1)}{\pi_2 u_1^B(2)} \tag{5}$$

Market clearance

$$x_1^A\left(p_s, e_0^A\right) + x_1^B\left(p_s, e_0^B + p_s e_{1s}^B\right) = e_{1s}^B \tag{6}$$

implying:

$$p_s = p\left(e_{1s}^B\right) \tag{7}$$

Impact of deviation from production decision

$$\frac{de_{12}}{de_{11}} = -\frac{F_1}{F_2} \tag{8}$$

$$x_{1s}^A = -\left(x_{1s}^B - e_{1s}^B\right) \tag{9}$$

$$\begin{aligned}\frac{d}{de_{11}^B}\sum_s \pi_s u^A(s) &= -\pi_1 u_0^A(1)\, x_{11}^A p'\left(e_{11}^B\right) - \pi_2 u_0^A(2)\, x_{12}^A p'\left(e_{12}^B\right)\frac{de_{12}}{de_{11}} \\ &= -\pi_1 u_0^A(1)\left[x_{11}^A p'\left(e_{11}^B\right) + \frac{\pi_2 u_0^A(2)}{\pi_1 u_0^A(1)} x_{12}^A p'\left(e_{12}^B\right)\frac{de_{12}}{de_{11}}\right]\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{d}{de_{11}^B}\sum_s \pi_s u^B(s) &= -\pi_1 u_0^B(1)\left[\left(x_{11}^B - e_{11}^B\right) p'\left(e_{11}^B\right) + \frac{\pi_2 u_0^B(2)}{\pi_1 u_0^B(1)}\left(x_{12}^B - e_{12}^B\right) p'\left(e_{12}^B\right)\frac{de_{12}}{de_{11}}\right] \\ &= \pi_1 u_0^B(1)\left[x_{11}^A p'\left(e_{11}^B\right) + \frac{\pi_2 u_0^B(2)}{\pi_1 u_0^B(1)} x_{12}^A p'\left(e_{12}^B\right)\frac{de_{12}}{de_{11}}\right]\end{aligned} \tag{11}$$

II. Change in production with redistribution

We now add redistribution in numeraire good, at the same level in both states of nature.

This changes market clearance to:

$$x_1^A\left(p_s, e_0^A - T\right) + x_1^B\left(p_s, e_0^B + T\right) = e_{1s}^B \tag{12}$$

implying:

$$p_s = p\left(e_{1s}^B, T\right) \tag{13}$$

Note that

$$\frac{\partial p_s}{\partial T} = \frac{\dfrac{\partial x_1^A}{\partial I} - \dfrac{\partial x_1^B}{\partial I}}{\dfrac{\partial x_1^A}{\partial p} - T_1\dfrac{\partial x_1^A}{\partial I} + \dfrac{\partial x_1^B}{\partial p} + \left(e_{1s}^B + T_1\right)\dfrac{\partial x_1^B}{\partial I}} \tag{14}$$

$$\frac{\partial p_s}{\partial T_1} = \frac{p_s\left\{\dfrac{\partial x_1^A}{\partial I} - \dfrac{\partial x_1^B}{\partial I}\right\}}{\dfrac{\partial x_1^A}{\partial p} - T_1\dfrac{\partial x_1^A}{\partial I} + \dfrac{\partial x_1^B}{\partial p} + \left(e_{1s}^B + T_1\right)\dfrac{\partial x_1^B}{\partial I}} \tag{15}$$

As long as the income derivatives of A and B are different, these are nonzero. Also the demand derivatives are evaluated at different prices and incomes in the different states.

Starting with zero transfers, consider a derivative change in the two transfers, satisfying (for some constant k).

$$dT_1 = kdT_0 \tag{16}$$

This implies that

$$\frac{dp_s}{dT_0} \equiv \frac{\partial p_s}{\partial T_0} + k\frac{\partial p_s}{\partial T_1} = \frac{\left(1+kp_s\right)\left\{\frac{\partial x_1^A}{\partial I} - \frac{\partial x_1^B}{\partial I}\right\}}{\frac{\partial x_1^A}{\partial p} - T_1\frac{\partial x_1^A}{\partial I} + \frac{\partial x_1^B}{\partial p} + \left(e_{1s}^B + T_1\right)\frac{\partial x_1^B}{\partial I}} \equiv \left(1+kp_s\right)\alpha_s \tag{17}$$

We want to evaluate the impact of a redistribution on expected utilities in equilibrium.

$$\begin{aligned}\frac{d}{dT_0}\sum_s \pi_s u^A(s) &= -\sum_s \pi_s\left(u_0^A(s) + ku_1^A(s)\right)\\ &\quad -\pi_1 u_0^A(1)\,x_{11}^A\frac{dp_1}{dT_0} - \pi_2 u_0^A(2)\,x_{12}^A\frac{dp_2}{dT_0}\\ &= -\pi_1 u_0^A(1)\left[\left(1+kp_1\right)\left(1+x_{11}^A\alpha_1\right) + \frac{\pi_2 u_0^A(2)}{\pi_1 u_0^A(1)}\left(1+kp_2\right)\left(1+x_{12}^A\alpha_2\right)\right]\end{aligned} \tag{18}$$

Similarly, using the same substitutions as in (11),

$$\frac{d}{dT_0}\sum_s \pi_s u^B(s) = \pi_1 u_0^B(1)\left[\left(1+kp_1\right)\left(1+x_{11}^A\alpha_1\right) + \frac{\pi_2 u_0^B(2)}{\pi_1 u_0^B(1)}\left(1+kp_2\right)\left(1+x_{12}^A\alpha_2\right)\right] \tag{19}$$

Generically we have different prices and demands in the two states and different marginal rates of substitution for the two agents. The aim is to find a constant, k, so that the changes in transfers leave both of them better off or both worse off (in which case we reverse the direction of transfers). This may be possible – this model does not fit the Inefficiency Theorem. Contrasting (18) and (19) to (10) and (11), we have an extra degree of freedom in seeking a Pareto gain.

For a Pareto gain, we need to find a value of k such that (18) and (19) are both positive or both negative (calling for a reversal of the direction of redistribution). This requires

$$\frac{\pi_2 u_0^A(2)}{\pi_1 u_0^A(1)} < -\frac{(1+kp_2)(1+x_{12}^A\alpha_2)}{(1+kp_1)(1+x_{11}^A\alpha_1)} < \frac{\pi_2 u_0^B(2)}{\pi_1 u_0^B(1)} \tag{20}$$

or

$$\frac{\pi_2 u_0^A(2)}{\pi_1 u_0^A(1)} > -\frac{(1+kp_2)(1+x_{12}^A\alpha_2)}{(1+kp_1)(1+x_{11}^A\alpha_1)} > \frac{\pi_2 u_0^B(2)}{\pi_1 u_0^B(1)} \tag{21}$$