14.126 Game theory
Problem Set 4

The due date for this assignment is Friday April 23. Please quote your sources.

1. For any set S, let X be the set of all open subsets of S. Show that $(X, \supseteq)$ is a complete lattice. What are the join and meet operators?

2. Prove the following statements.

   (a) If f and g are supermodular, so are $f + g$ and af for $a > 0$.

   (b) If f and g are supermodular, isotone and nonnegative, then fg is supermodular.

   (c) If $f : \Theta \times X \to \mathbb{R}$, where

   - X is a lattice,
   - $\theta \in \Theta$ is not known,
   - $f(\theta, \cdot) : X \to \mathbb{R}$ is supermodular for each $\theta \in \Theta$,

   then $E[f] : X \to \mathbb{R}$ is supermodular, where E is an expectation operator on Θ.

3. Alice wants to submit a proxy bid b for an object. The price of object is p, which is uniformly distributed on $[0, \bar{p}]$. Alice does not know p. If $b \geq p$, she gets the object and pays p; otherwise she does not get the object and does not pay anything. The value of the object for Alice is

$$v = V \exp\left(\sum_{i=0}^{\infty} \alpha_i \sum_{j_i=1}^{J_i} X_{i,j}\right)$$

where $V \geq 0$ and $\alpha_i \geq 0$ are known constants and $(X_{i,j})$ are bounded, independently distributed random variables, such that v is bounded by $\bar{p}$ from above and independent of p. Moreover, for each i, $X_{i,1}, \ldots, X_{i,J_i}$ are identically distributed. Here, V represents the ex-ante value of the object, and each i is an attribute of the object. Alice does not know the values of $(X_{i,j})$, but before submitting her bid, for each i, she can learn the values of n_i variables in $X_{i,1}, \ldots, X_{i,J_i}$ by incurring cost $C_i(n_i; c_i)$ where C_i is supermodular and c_i is a cost parameter. Her total payoff is

$$U(n, b; v, p) = \begin{cases} v - p - \sum_{i=0}^{\infty} C_i(n_i; c_i) & \text{if } b \geq p \\ -\sum_{i=0}^{\infty} C_i(n_i; c_i) & \text{otherwise.} \end{cases}$$

Write $n = (n_i)_{i=0}^{\infty} \geq n' = (n'_i)_{i=0}^{\infty}$ iff $n_i \geq n'_i$ for each i.

   (a) What is the optimal bid b^*, as a function of what she learns?

   (b) Show that the set n^* of optimal solutions is a lattice.

   (c) Show that n^* weakly increasing in V, $1/\bar{p}$, α_i and $-c_i$ for each i. Briefly interpret the comparative statics (focusing on the largest solution).

4. Consider a Bertrand oligopoly with n firms in which each firm i chooses both its price p_i and the advertisement level r_i. For each firm i, the demand $Q_i(\theta, p, r_i)$ for its product is supermodular, decreasing in p_i and increasing in all other variables, where θ is a known demand parameter. The marginal cost of i is c_i so that the payoff of i is

$$U_i(p, r) = (p_i - c_i) Q_i(\theta, p, r_i) - \gamma_i r_i^2/2,$$

where $\gamma_i \geq 0$ is a cost parameter for advertisement. The argument of U_i emphasizes that the strategy profile is a pair of a price vector p and and advertisement vector r. The set of possible prices p_i is $[c_i, \bar{p}_i]$ for some $\bar{p}_i > c_i$, and the set of advertisement levels is $[0, 1]$.

(a) Show that there exists equilibria (p^*, r^*) and (p^{**}, r^{**}) such that $(p^*, r^*) \geq (p, r) \geq (p^{**}, r^{**})$ for every rationalizable strategy profile (p, r).

(b) How do (p^*, r^*) and (p^{**}, r^{**}) vary with respect to θ, γ, and c? If they are monotone with respect to a parameter, prove it; otherwise provide an example.

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