Review of Basic Concepts: Normal form

14.126 Game Theory

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Road Map

- Normal-form Games
- Dominance & Rationalizability
- Nash Equilibrium
  - Existence and continuity properties
- Bayesian Games
  - Normal-form/agent-normal-form representations
  - Bayesian Nash equilibrium—equivalence to Nash equilibrium, existence and continuity

Normal-form games

- A (normal form) game is a triplet (N, S, u):
  - $N = \{1, \ldots, n\}$ is a (finite) set of players.
  - $S = S_1 \times \ldots \times S_n$ where S_i is the set of pure strategies of player i.
  - $u = (u_1,\ldots,u_n)$ where $u_i : S \to \mathbb{R}$ is player i's vNM utility function.
- A normal form game is finite if S and N are finite.
- The game is common knowledge.

Mixed Strategies, beliefs

- $\Delta(X)$ = Probability distributions on X.
- $\Delta(S_i)$ = Mixed strategies of player i.
- Independent strategy profile:

$$\sigma = \sigma_1 \times \ldots \times \sigma_n \in \Delta(S_1) \times \ldots \times \Delta(S_n)$$

- correlated strategy profile:

$$\sigma \in \Delta(S)$$

- $\Delta(S_{-i})$ = possible conjectures of player i (beliefs about the other players' strategies). $[\sigma_{-i} \in \Delta(S_{-i})]$
  - A player may believe that the other players' strategies are correlated!
- Expected payoffs:

$$u_i(\sigma) = E_\sigma(u_i) = \Sigma_{s \in S}\sigma(s)u_i(s)$$

Rationality & Dominance

- Player i is rational if he maximizes his expected payoff given his belief.
- s_i^* is a best reply to a belief σ_{-i} iff
  $\forall s_i \in S_i: u_i(s_i^*,\sigma_{-i}) \geq u_i(s_i,\sigma_{-i})$.
- $B_i(\sigma_{-i})$ = best replies to σ_{-i}.
- σ_i strictly dominates s_i iff
  $\forall s_{-i} \in S_{-i}: u_i(\sigma_i,s_{-i}) > u_i(s_i,s_{-i})$.
- σ_i weakly dominates s_i iff
  $\forall s_{-i} \in S_{-i}: u_i(\sigma_i,s_{-i}) \geq u_i(s_i,s_{-i})$ with a strict inequality.

Theorem: In a finite game, s_i^* is never a best reply to a (possibly correlated) conjecture σ_{-i} iff s_i^* is strictly dominated (by a possibly mixed strategy).

Proof of Theorem

- Let
  - $S_{-i} = \{s_{-i}^1,\ldots, s_{-i}^m\}$,
  - $u_i(s_i,.) = (u_i(s_i,s_{-i}^1),\ldots, u_i(s_i,s_{-i}^m))$
  - $U = \{u_i(s_i,.)|s_i \in S_i\}$
  - Co(U) = convex hull of U
    $= \{u_i(\sigma_i,.)|\sigma_i \in \Delta(S_i)\}$
- (=>) Assume $s_i^* \in B_i(\sigma_{-i})$.
  $\Rightarrow \forall s_i, u_i(s_i^*,\sigma_{-i}) \geq u_i(s_i,\sigma_{-i})$
  $\Rightarrow \forall \sigma_i, u_i(s_i^*,\sigma_{-i}) \geq u_i(\sigma_i,\sigma_{-i})$
  $\Rightarrow$ No σ_i strictly dominates s_i^*.
- **SHT:** Let C and D be non-empty, disjoint subsets of $\mathbb{R}^m$ with C closed. Then, $\exists r \in \mathbb{R}^m \setminus \{0\} : \forall x \in \text{cl}(D)\ \forall y \in C$, $r \cdot x \geq r \cdot y$.
- (<=) Define
  $D = \{x \in \mathbb{R}^m | x_k > u_i(s_i^*,s_{-i}^k)\ \forall k\}$.
- Assume s_i^* is not strictly dominated.
- Co(U) and D are disjoint.
- By SHT, $\exists r: \forall \sigma_i$,
  $u_i(s_i^*,\sigma_{-i}) \geq u_i(\sigma_i,\sigma_{-i})$
  where $\sigma_{-i}(s_{-i}^k) = r^k/(r^1+\ldots+ r^m)$

Iterated strict dominance & Rationalizability

- $S^0 = S$
- $S_i^k = B_i(\Delta(S_{-i}^{k-1}))$
- (Correlated) Rationalizable strategies:

$$S_i^\infty = \bigcap_{k=0}^{\infty} S_i^k$$

- Independent rationalizability: $s_i \in S_i^k$ iff $s_i \in B_i(\Pi_{j \neq i}\sigma_j)$ where $\sigma_j \in \Delta(S_j^{k-1})$ $\forall j$.
- σ_i is rationalizable iff $\sigma_i \in B_i(\Delta(S_{-i}^\infty))$.

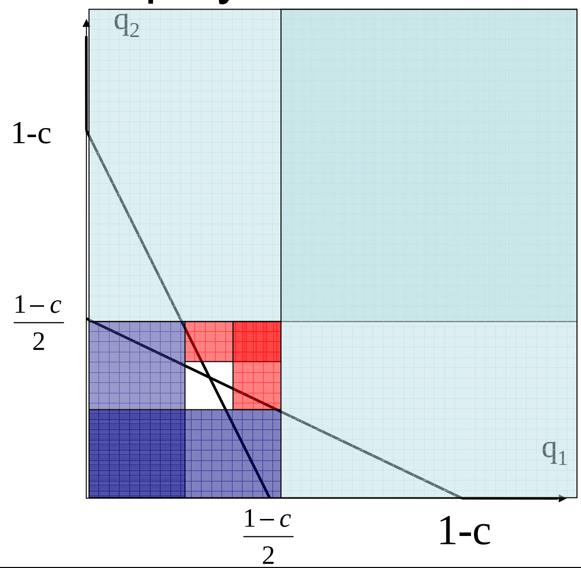
Theorem (fixed-point definition): S^∞ is the largest set $Z_1 \times \ldots \times Z_n$ s.t. $Z_i \subseteq B_i(\Delta(Z_{-i}))$ for each i. (s_i is rationalizable iff $s_i \in Z_i$ for such $Z_1 \times \ldots \times Z_n$.)

Foundations of rationalizability

- If the game and rationality are common knowledge, then each player plays a rationalizable strategy.
- Each rationalizable strategy profile is the outcome of a situation in which the game and rationality are common knowledge.
- In any "adaptive" learning model the ratio of players who play a non-rationalizable strategy goes to zero as the system evolves.

Rationalizability in Cournot Duopoly

Simultaneously, each firm $i \in \{1,2\}$ produces q_i units at marginal cost c, and sells it at price $P = \max\{0, 1-q_1-q_2\}$.



Rationalizability in Cournot duopoly

- If i knows that $q_j \le q$, then $q_i \ge (1-c-q)/2$.
- If i knows that $q_j \ge q$, then $q_i \le (1-c-q)/2$.
- We know that $q_j \ge q^0 = 0$.
- Then, $q_i \le q^1 = (1-c-q^0)/2 = (1-c)/2$ for each i;
- Then, $q_i \ge q^2 = (1-c-q^1)/2 = (1-c)(1-1/2)/2$ for each i;
- …
- Then, $q^n \le q_i \le q^{n+1}$ or $q^{n+1} \le q_i \le q^n$ where $q^{n+1} = (1-c-q^n)/2 = (1-c)(1-1/2+1/4-\ldots+(-1/2)^n)/2$.
- As $n \to \infty$, $q^n \to (1-c)/3$.

Nash Equilibrium

- The following are equivalent:
  - $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a **Nash Equilibrium**
  - $\forall i$, $\sigma_i^* \in B_i(\sigma_{-i}^*)$, where B_i contains mixed best replies
  - $\forall i$, $\forall s_i \in S_i$: $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$,
  - $\forall i$, $\text{supp}(\sigma_i^*) \subseteq B_i(\sigma_{-i}^*)$.
- Aumann & Brandenburger: In a 2-person game, if game, rationality, and conjectures are all mutually known, then the conjectures constitute a Nash equilibrium.
- For n>2 players, we need common prior assumption and common knowledge of conjectures.
- Steady states of any adaptive learning process are Nash equilibria.

Existence and continuity

- For any correspondence $F : X \to 2^Y$, where X compact and Y bounded, F is upper-hemicontinuous iff F has closed graph:

$$[x_m \to x \ \& \ y_m \to y \ \& \ y_m \in F(x_m)] \Rightarrow y \in F(x).$$

- Berge's Maximum Theorem (existence and continuity of individual optimum): Assume $f : X \times Z \to Y$ is continuous and X, Y, Z are compact. Let

$$F(x) = \arg\max_{z \in Z} f(x,z).$$

  Then, F is non-empty, compact-valued, and upper-hemicontinuous.
- Kakutani's Fixed-point theorem: Let X be a convex, compact subset of $\mathbb{R}^m$ and let $F: X \to 2^X$ be a non-empty, convex-valued correspondence with closed graph. Then, there exists $x \in X$ such that $x \in F(x)$.

Existence of Nash Equilibrium

Theorem: Let each S_i be a convex, compact subset of a Euclidean space and each u_i be continuous in s and quasi-concave in s_i. Then, there exists a Nash equilibrium $s \in S$.

Corollary: Each finite game has a (possibly mixed) Nash equilibrium σ^*.

Proof of corollary: Each $\Delta(S_i) \subseteq \mathbb{R}^m$ is convex and compact. Each $u_i(\sigma)$ is continuous, and linear in σ_i. Then, the game with strategy spaces $\Delta(S_i)$ has a NE $\sigma^* \in \Delta(S_1) \times \ldots \times \Delta(S_n)$.

Proof of Existence Theorem

- Let $F : S \rightarrow 2^S$ be the "best reply" correspondence:

$$F_i(s) = B_i(s_{-i})$$

- By the Maximum Theorem, F is non-empty and has closed graph.
- By quasi-concavity, F is convex valued.
- By Kakutani fixed-point theorem, F has a fixed point: $s^* \in F(s^*)$.
- s^* is a Nash equilibrium.

Upper-hemicontinuity of NE

- X, S are compact metric spaces
- $u^x(s)$ is continuous in $x \in X$ and $s \in S$.
- NE(x) is the set of Nash equilibria of (N,S,u^x).
- PNE(x) is pure Nash equilibria of (N,S,u^x).

Theorem: NE and PNE are upper-hemicontinuous.

Corollary: If S is finite, NE is non-empty, compact-valued, and upper-hemicontinuous.

Proof:

- $\Delta(S_i)$ is compact and $u^x(\sigma)$ is continuous in (x,σ).
- Suppose: $x_m \to x$, $\sigma^m \in \text{NE}(x_m)$, $\sigma^m \to \sigma \notin \text{NE}(x)$.
- $\exists i, s_i$: $u^x(s_i,\sigma_{-i}) > u^x(\sigma)$.
- $u^{x_m}(s_i,\sigma_{-i}^m) > u^{x_m}(\sigma^m)$ for large m.

Bayesian Games

- A Bayesian game is a list (N, A, Θ, T, u, p):
  - $N = \{1, \ldots, n\}$ is a (finite) set of players;
  - $A = A_1 \times \ldots \times A_n$; A_i is the set of actions of i;
  - Θ is the set of payoff relevant parameters;
  - $T = T_1 \times \ldots \times T_n$; T_i is the set of types of i;
  - $u = (u_1,\ldots,u_n)$; $u_i : \Theta \times A \to \mathbb{R}$ is i's vNM utility function;
  - $p \in \Delta(\Theta \times T)$ is a common prior.
- A Bayesian game is a list (N, A, Θ, T, u, p) as above except $u_i : \Theta \times T \times A \to \mathbb{R}$.
- A Bayesian game is a list (N, A, T, u, p) as above except $u_i : T \times A \to \mathbb{R}$.

Fact: All three formulations are equivalent (as long as you know what you are doing).

Fact: We can replace p with $p_1,\ldots, p_n$, dropping CPA.

Normal-form representations

- Given a Bayesian game $\Gamma=(N,A,\Theta,T,u,p)$,
- Normal Form: $G(\Gamma)= (N,S,U)$:
  - S_i = {functions s_i: $T_i \to A_i$}
  - $U_i(s) = E_p[u_i(\theta,s_1(t_1),\ldots,s_n(t_n))]$.
- Agent-Normal Form: $AG(\Gamma)= (\underline{N},S,U)$:

$$\underline{N} = T_1 \cup \cdots \cup T_n$$

$$S_{t_i} = A_i \text{ for each } t_i \in T_i$$

$$U_{t_i}(s) = E_p[u_i(\theta, s_1(t_1),\ldots,s_n(t_n)) \mid t_i]$$

Bayesian Nash equilibrium

Definition: $\sigma^* = (\sigma_1^*,\ldots,\sigma_n^*)$ is a **Bayesian Nash Equilibrium** iff for each i, t_i,

$$\sigma_i^*(a_i \mid t_i) > 0 \Rightarrow a_i \in \arg\max_{\bar{a}_i} E_p[u_i(\theta, \bar{a}_i, \sigma_{-i}^*(a_{-i} \mid t_{-i})) \mid t_i]$$

Fact: σ^* is a Bayesian Nash equilibrium of Γ iff the profile $\sigma^*(\cdot|t_i)$, $t_i \in T_i$, $i \in N$ is a Nash equilibrium of $AG(\Gamma)$.

Fact: If σ^* is a Bayesian Nash equilibrium of Γ, then σ^* is a Nash equilibrium of $G(\Gamma)$. If $p(t_i)>0$ for each t_i, the converse is also true.

Existence of BNE

Consider $\Gamma=(N,A,\Theta,T,u,p)$ with finite N and T.

Theorem: If

- each A_i is compact and convex
- each u_i is bounded, continuous in a, concave in a_i,

then Γ has a pure Bayesian Nash equilibrium.

Proof: $AG(\Gamma)$ has a pure Nash equilibrium.

Corollary: If A is finite, Γ has a (possibly mixed) Bayesian Nash equilibrium.

Upper-hemicontinuity of BNE

- A, T finite and Θ, X compact.
- $u_i^x(\theta,a)$ continuous in (x,θ,a)
- BNE(x) Bayesian NE of $\Gamma^x = (N,A,\Theta,T,u^x,p)$.
- BNE(p) Bayesian Nash equilibria of (N,A,Θ,T,u,p).

Theorem: BNE is upper-hemicontinuous.

Proof: BNE(x) = NE($AG(\Gamma^x)$).

Theorem: Assume $p(t_i) > 0$ $\forall p\in P$, $\forall t_i\in T_i$, for compact $P \subseteq \Delta(\Theta\times T)$. BNE($p$) is upper-hemicontinuous on P.

Proof: $U_i(s;p) = E_p[u_i(\theta,s_1(t_1),\ldots,s_n(t_n))]$ is continuous; BNE(p) = NE($G((N,A,\Theta,T,u,p))$).