14.126 Lecture Notes on Supermodular Games

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A common exercise in economics is to understand how a particular outcome varies qualitatively varies with a particular parameter, e.g., whether income tax increases the investment level in equilibrium. When one can answer such a question, the is often driven by a supermodularity (or complementarity) assumption. In these lectures, I will formally introduce supermodular games and the mathematics used in their analysis.

Complementarities are expresses both in terms of constraints and payoff functions. In terms of constraints, two activities are complementary if doing one activity more does not reduce the possible activity level for the other activity. This is mathematically captured by *lattices*. In terms of payoffs, two activities are complementary if doing one activity more increases the marginal benefit of doing the other. This is mathematically captured by supermodular payoff functions.

The main result in these lecture will establish the structure of the maximizing solutions and monotone comparative statics under complementarity both in terms of constraints and payoff function. For the individual decision problems, the result establishes that the set of solutions is a lattice and weakly increasing in complementary parameters. For games, the result establishes that there are extremal pure strategy equilibria that bound all rationalizable strategies, and the extremal equilibria are monotone with respect to complementary parameters.

1 Example

As a concrete example of the forthcoming example consider the Diamond's search model. There is a continuum of players. Each player i puts effort $a_i \in [0, 1]$, incurring a cost of $a_i^2/2$. Let $\bar{a}_{-i}$ be the average effort level for the players other than i. The probability that i finds a match is $a_i g(\bar{a}_{-i})$ for some increasing, continuous function $g : [0, 1] \to g\,[0, 1]$ with $g\,(0) = 0$ and $g\,(1) = 1$. Let the payoff from match be $\theta \geq 0$. Then, the expected payoff of player i is

$$U_i(a) = \theta a_i g(\bar{a}_{-i}) - a_i^2/2.$$

We have strategic complementarity (i.e. the complementarity between the strategies) because an increase in $\bar{a}_{-i}$ always results in a (weakly) increase in the $\partial U_i/\partial a_i$. This results in an increasing best-response function

$$BR_i\,(a_{-i}) = \theta g(\bar{a}_{-i}).$$

Similarly, there is a complementarity between the search level a_i and the value θ of match: $\partial^2 U_i/\partial a_i \partial \theta = g\,(\bar{a}_{-i}) \geq 0$. Once again the best response is increasing in θ.

Consider the equilibria of the above game. Note that, since the best response function is increasing and the payoffs are symmetric, every equilibrium is symmetric. Equilibria are then characterized by the intersection of the graph of g with the diagonal, as in Figure 1.

In this figure, there are three equilibria, all of the equilibria are ordered. Among these the smallest equilibrium, located at the origin, and the largest equilibrium are stable, the middle equilibrium is unstable. While the number of equilibria depends on the shape of g, the equilibria will always be ordered (because g and the diagonal are increasing), and we will have extremal equilibria. The latter is indeed a general property of supermodular games.

In order to investigate how the equilibrium search levels vary by the complementary parameter of the value of the match, increase θ to a higher level θ'. Since this corrosions to scaling up the best response function (by θ'/θ), the new equilibria are formed as in the figure. The smallest equilibrium remains at zero (weakly increasing). The largest equilibrium moves up. These changes are intuitive in that we would expect the players to search more when the match is more valuable. This will indeed be generally true for all supermodular games. Note however that the middle equilibrium, which is unstable,

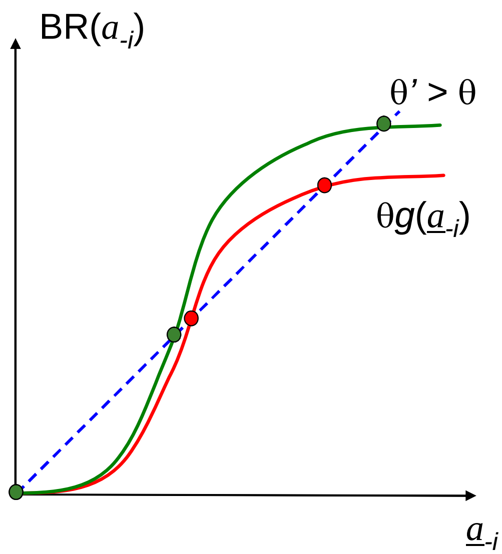

Figure 1: Equilibria in Diamond's search model

decreases, so that the players search less when the match is more valuable. This shows that the intuition is true only for the extreme equilibria, and one should keep this counterexample in mind throughout.

Finally, note that the largest equilibrium moves more than individual best responses, i.e., $a_i^*[\theta'] > BR_i(a_i^*[\theta], \theta')$, where $a_i^*[\theta]$ and $a_i^*[\theta']$ are the equilibrium strategies under θ and θ', respectively. That is, there is a multiplier effect. Although I will not explore this issue in these notes, this is also true under broad conditions.

2 Lattices and Supermodularity

This section presents the basic concepts in lattice theory.

2.1 Lattices

A partially-ordered set is said to be lattice if each doubleton subset has greatest lower bound (inf) and smallest upper bound (sup).

Definition 1 *A partially-ordered set* $(X, \geq)$ *is said to be* lattice *iff for all* $x, y \in X$,

$$x \vee y \equiv \inf\{z \in X | z \geq x, z \geq y\} \in X$$
$$x \wedge y \equiv \sup\{z \in X | x \geq z, y \geq z\} \in X.$$

Here, operators $\vee$ and $\wedge$ are called *join* and *meet*, respectively. Note that $x \vee y \in X$ is such that $x \vee y \geq x$, $x \vee y \geq y$, and moreover if $z \geq x$ and $z \geq y$, then $z \geq x \vee y$. That is, $x \vee y$ is the smallest upper bound for $\{x, y\}$. Similarly, $x \wedge y$ is the greatest lower bound for $\{x, y\}$ in the sense that $x \geq x \wedge y$, $y \geq x \wedge y$, and if $x \geq z$ and $y \geq z$, then $x \wedge y \geq z$.

Example 1 *Let* $X = 2^S$ *be all the set of all subsets of a set* S, *and order* X *by set inclusion, i.e.,* $A \geq B \iff A \supseteq B$. *For any* $A, B \in X$, *note that* $A \cup B \supseteq A$, $A \cup B \supseteq B$ *and if* $C \supseteq A$ *and* $C \supseteq B$, *then* $C \supseteq A \cup B$. *Therefore,* $A \vee B = A \cup B \in X$. *Similarly,* $A \wedge B = A \cap B \in X$. *Therefore,* $(X, \supseteq)$ *is a lattice.*

Example 2 *Endow* $\mathbb{R}^n$ *with the usual coordinate-wise order:*

$$(x_1, \ldots, x_n) \geq (y_1, \ldots, y_n) \iff x_i \geq y_i \; \forall i.$$

$(\mathbb{R}^n, \geq)$ *is a lattice with*

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$$
$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}).$$

Definition 2 *A lattice* $(X, \geq)$ *is said to be complete if for every* $S \subseteq X$, *a greatest lower bound inf*(S) *and a least upper bound sup*(S) *exist in* X, *where* $\inf(\varnothing) = \sup(X)$ *and* $\sup(\varnothing) = \inf(X)$.

Note that, in the above examples, $(2^S, \supseteq)$ is complete because for any family $A_\alpha \subseteq S$, $\vee_\alpha A_\alpha = \cup_\alpha A_\alpha \in 2^S$ and $\wedge_\alpha A_\alpha = \cap_\alpha A_\alpha \in 2^S$. On the other hand, $(\mathbb{R}^n, \geq)$ is not complete because $\sup(\mathbb{R}^n)$ does not exist.

2.2 Strong Set Order and Sublattices

Given a lattice $(X, \geq)$, one can extend the order $\geq$ to subsets of X as follows.

Definition 3 (Strong Set Order) *Given any lattice $(X, \geq)$, for any $A, B \subseteq X$,*

$$A \geq B \iff [x \geq y \qquad \forall x \in A, y \in B].$$

With the usual order on $\mathbb{R}$, note that $\{1,2,3,4\} \geq \{0,1,2,3\}$ but $\{1,2,3,4\} \ngeq \{-0.5, 0.5, 1, 5, 2, 5\}$. This is a very strong notion of order as it implies many other natural orders on sets. For example, if $A \geq B$, then $\max A \geq \max B$ and $\min A \geq \min B$.

A lattice may have a subset that is a lattice in itself according to original order. Such subsets are called sublattices.

Definition 4 *Given any lattice $(X, \geq)$, any $S \subseteq X$ is said to be* sublattice *if for any $x, y \in S$, $x \vee y \in S$ and $x \wedge y \in S$.*

The following gives an equivalent definition for sublattices.

Fact 1 *Given any lattice $(X, \geq)$ and any $S \subseteq X$, S is a sublattice iff $S \geq S$.*

For example, under the usual order, $S = \{(x_1, x_2) : x_1 + x_2 \leq 1\}$ is not a sublattice of $\mathbb{R}^2$ because $(1,0) \vee (0,1) = (1,1) \notin S$. On the other hand, $[0,1]^2$ and $S' = \{(x_1, x_2) : x_1 - x_2 \leq 1\}$ are sublattices.

2.3 Functions on Lattices — Supermodularity

I will next introduce important properties of functions to or from lattices. The first property is an elementary monotonicity property, requiring that the order is preserved.

Definition 5 *Given any partially ordered sets $(T, \geq)$ and $(X, \geq)$, a function $f : T \to X$ is said to be* isotone *(or* weakly increasing*) if*

$$t \geq t' \Rightarrow f(t) \geq f(t').$$

Throughout the lecture, we will take t to be a parameter and investigate how it effects the outcomes according to a solution concept. Since our solution concepts, such as argmax and Nash equilibrium, are set valued the above definition will be often applied to set-valued functions. Note that, for any lattice $(Y, \geq)$, $(2^Y, \geq)$ is a partially ordered set (with the strong set order).

The second property formalizes the idea of complementarity in terms of functions:

Definition 6 *Given any lattice $(X, \geq)$, a function $f : X \to \mathbb{R}$ is said to be* supermodular *if for all $x, y \in X$,*

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

A function f is said to be submodular if $-f$ is supermodular.

Note that if X is linearly ordered (as R), then every function $f : X \to R$ is supermodular as the above inequality is vacuously satisfied as equality.

When $X = X_1 \times X_2$, ordered coordinate-wise, supermodularity captures the idea of complementarity between X_1 and X_2 precisely. Indeed, if we take $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $x_1 \geq y_1$ with $y_2 \geq x_2$, we have $x \vee y = (x_1, y_2)$ and $x \wedge y = (y_1, x_2)$.Then, we can write the inequality in the definition of supermodularity as

$$f(x_1, y_2) - f(x_1, x_2) \geq f(y_1, y_2) - f(y_1, x_2).$$

That is, the marginal contribution of increasing the second input from x_2 to y_2 increases when we increase the first input from y_1 to x_1. That is, marginal contribution of an input is increasing with the other input, capturing the usual meaning of complementarity (as in production theory).

One can also withe the above inequality as a condition on the mixed differences:

$$[f(x_1, y_2) - f(x_1, x_2)] - [f(y_1, y_2) - f(y_1, x_2)] \geq 0.$$

This condition reduces to a usual restriction on the cross-derivatives for smooth functions on $\mathbb{R}^2$:

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} \geq 0.$$

Supermodularity turns out to be closely related to the monotone comparative statistics, an ordinal property. Despite this, note that supermodularity is a cardinal property, as it is not preserved under monotone transformation, as the next example illustrates.

Example 3 *Let $X = \{0, 1\}^2$ and endow it with the usual order. Consider the following supermodular function:*

$$f(1,1) = 3, f(1,0) = f(0,1) = 1, f(0,0) = 0.$$

Note that $\sqrt{f}$ is not supermodular.

2.4 Increasing Differences and Supermodularity in Product Spaces

In game theoretical applications, the lattices are of the product form as the space of strategy profile is a product set. In such lattices supermodularity reduces to a simpler condition.

For a family of lattices $(X_1, \geq), \ldots, (X_n, \geq)$, let $X = X_1 \times \cdots \times X_n$ and endow X with the coordinate-wise order (i.e., $(x_1, \ldots, x_n) \geq (y_1, \ldots, y_n)$ iff $x_i \geq y_i$ for each i). For $x \in X$ and any i and j, define $x_{-ij} = (x_k)_{k \notin \{i,j\}}$. For any function $f : X \to \mathbb{R}$, define $f(\cdot|x_{-ij}) : X_i \times X_j \to \mathbb{R}$ by setting $f(x_i, x_j|x_{-ij}) = f(x_i, x_j, x_{-ij})$. Note that $f(\cdot|x_{-ij})$ is the restriction of f to vectors where the entries other than i and j are fixed at x_{-ij}.

Definition 7 *A function $f : X \to \mathbb{R}$ is said to* have increasing differences *iff for any $\left(i, j, x_{-ij}, x_i, x_i', x_j, x_j'\right)$,*

$$\left[x_i \geq x_i' \text{ and } x_j \geq x_j'\right] \Rightarrow f(x_i, x_j, x_{-ij}) - f(x_i', x_j, x_{-ij}) \geq f\left(x_i, x_j', x_{-ij}\right) - f\left(x_i', x_j', x_{-ij}\right).$$

That is, ceteris peribus, the marginal contribution of ith entry (obtained by changing x_i' to higher x_i) is higher when the jth entry is fixed at the higher level of x_j rather than x_j'. When $X = \mathbb{R}^n$, the condition of increasing differences can be called pair-wise supermodularity, because the above condition can be written as a supermodularity condition on function $f(\cdot|x_{-ij}) : X_i \times X_j \to \mathbb{R}$, defined by setting $(x_i, x_j|x_{-ij}) = f(x_i, x_j, x_{-ij})$. That is,

$$f\left((x_i, x_j) \vee \left(x_i', x_j'\right), x_{-ij}\right) - f\left((x_i, x_j), x_{-ij}\right) \geq f\left(\left(x_i', x_j'\right), x_{-ij}\right) - f\left((x_i, x_j) \wedge \left(x_i', x_j'\right), x_{-ij}\right).$$

Both the condition of increasing differences and pair-wise supermodularity are weaker forms of supermodularity as it is the restriction of supermodularity condition to a special set of cases. (Supermodularity is weakly stronger than pair-wise supermodularity, and pair-wise supermodularity is weakly stronger than increasing difference condition.) It turns out that supermodularity can be decomposed into increasing differences and supermodularity within each X_i.

The following lemma is a main step towards establishing this fact. Its proof also exhibit a common technique of using telescopic equation.

Lemma 1 *If f has increasing differences and $x_j \geq y_j$ for each j, then for every i,*

$$f(x_i, x_{-i}) - f(y_i, x_{-i}) \geq f(x_i, y_{-i}) - f(y_i, y_{-i}).$$

Proof. Take $i = 1$ without loss of generality. Then,

$$\begin{aligned} f(x_1, x_{-1}) - f(x_1, y_{-1}) &= \sum_{j>1} \left[f\left(x_1, \ldots, x_{j-1}, x_j, y_{j+1}, \ldots, y_n\right) - f\left(x_1, \ldots, x_{j-1}, y_j, \ldots, y_n\right) \right] \\ &\geq \sum_{j>1} \left[f\left(y_1, \ldots, x_{j-1}, x_j, y_{j+1}, \ldots, y_n\right) - f\left(y_1, \ldots, x_{j-1}, y_j, \ldots, y_n\right) \right] \\ &= f(y_1, x_{-1}) - f(y_1, y_{-1}). \end{aligned}$$

Here, the first and the last equalities are telescopic equations, writing the whole difference as a sum of one step changes. The inequality is by increasing differences: for any j, by increasing differences between 1 and j,

$$\begin{aligned} & f\left(x_1, \ldots, x_{j-1}, x_j, y_{j+1}, \ldots, y_n\right) - f\left(x_1, \ldots, x_{j-1}, y_j, \ldots, y_n\right) \\ \geq\ & f\left(y_1, \ldots, x_{j-1}, x_j, y_{j+1}, \ldots, y_n\right) - f\left(y_1, \ldots, x_{j-1}, y_j, \ldots, y_n\right), \end{aligned}$$

and one obtains the inequality by summing up both sides. Of course, this is equivalent to the statement in the lemma. ■

Lemma extends the increasing differences condition from comparison of two entries to the comparison of two vectors, establishing a (seemingly) stronger increasing difference condition. This further implies that, in product spaces, supermodularity can be decomposed into increasing differences and supermodularity within each X_i, as established next.

Proposition 1 *For any product lattice $(X, \geq)$ (with $X = X_1 \times \cdots \times X_n$ and coordinate-wise order) and for any function $f : X \to \mathbb{R}$, f is supermodular if and only if*

1. *f has increasing differences and*

2. *f is supermodular within X_i for each i (i.e.,*

$$f\left(x_i \vee y_i, x_{-i}\right) + f\left(x_i \wedge y_i, x_{-i}\right) \geq f\left(x_i, x_{-i}\right) + f\left(y_i, x_{-i}\right)$$

   for all $x_i, y_i \in X_i$ and $x_{-i} \in X_{-i}$).

Proof. Supermodularity implies increasing differences (1) and supermodularity within each coordinate (2) by definition. To prove the converse, take any $x, y \in X$ and assume the conditions (1) and (2) in the proposition. Then,

$$
\begin{aligned}
& f(x \vee y) - f(y) \\
= & \sum_{i=1}^{n} \left[f(x_1 \vee y_1, \ldots, x_{i-1} \vee y_{i-1}, x_i \vee y_i, y_{i+1}, \ldots, y_n) - f(x_1 \vee y_1, \ldots, x_{i-1} \vee y_{i-1}, y_i, y_{i+1}, \ldots, y_n) \right] \\
\geq & \sum_{i=1}^{n} \left[f(x_1 \vee y_1, \ldots, x_{i-1} \vee y_{i-1}, x_i, y_{i+1}, \ldots, y_n) - f(x_1 \vee y_1, \ldots, x_{i-1} \vee y_{i-1}, x_i \wedge y_i, y_{i+1}, \ldots, y_n) \right] \\
\geq & \sum_{i=1}^{n} \left[f(x_1, \ldots, x_{i-1}, x_i, x_{i+1} \wedge y_{i+1}, \ldots, x_n \wedge y_n) - f(x_1, \ldots, x_{i-1}, x_i \wedge y_i, x_{i+1} \wedge y_{i+1}, \ldots, x_n \wedge y_n) \right] \\
= & f(x) - f(x \wedge y).
\end{aligned}
$$

Here, the first and the last equalities are telescopic equations; the first inequality is by (2), and the second inequality is by Lemma 1, which follows from increasing differences (1). Of course, each inequality is obtained by summing up the inequalities over i. ∎

An immediate corollary to the proposition is that supermodularity reduces to pair-wise supermodularity on $\mathbb{R}^n$:

Corollary 1 *for any $f : \mathbb{R}^n \to \mathbb{R}$, the following are equivalent:*

1. *f is supermodular;*
2. *f has increasing differences;*
3. *f is pair-wise supermodular.*

Proof. Since $\mathbb{R}$ is linearly ordered, every function is supermodular on $\mathbb{R}$. The corollary then follows from Proposition 1 and the definitions. ∎

Hence, supermodularity is a pair-wise concept on $\mathbb{R}^n$. This is because it reflects the pair-wise concept of complementarity. As mentioned in the introduction, as stated in terms of constraints, complementarity is also captured by sublattices. It turns out that sublattices can be reduced to pair-wise constraints on $\mathbb{R}^n$.

Proposition 2 *Let X be a sublattice of $\mathbb{R}^n$ (under coordinate-wise order). For all i, j, define*

$$
\begin{aligned}
C_{ij} &= \{(x_i, x_j) \mid (x_i, x_j, x_{-ij}) \in X \text{ for some } x_{-ij}\} \\
S_{ij} &= C_{ij} \times \prod_{k \neq i,j} \mathbb{R}.
\end{aligned}
$$

Then,

$$X = \bigcap_{i,j} S_{ij}.$$

That is, a sublattice X can be written as a set of pair-wise constraints: $x \in X$ iff

$$(x_i, x_j) \in C_{i,j} \qquad \forall i, j.$$

This limits the applicability of lattice theory and the analyses in these lectures substantially because many important constraints cannot be stated as sublattices. For example, when there are three or more goods, a budget set $\{x| \, (x - \bar{x}) \cdot p \leq 0\}$ cannot be a lattice under usual order or its reverse. (Some of the end results here are extended to allow such sets in later work.)

2.5 Order Topology and Continuity

The order in a lattice induces relevant concepts of continuity and convergence. I will conclude this section by describing these concepts, which will be used in the analyses of supermodular games.

Consider a complete lattice $(X, \geq)$. Consider any monotone sequence x_n in X. Since $(X, \geq)$ is complete,

$$\sup \{x_n | n \in \mathbb{N}\} \text{ and } \inf \{x_n | n \in \mathbb{N}\}$$

exist. For any weakly increasing sequence x_n (with $x_{n+1} \geq x_n$ for all n), it is natural to think that x_n converges to $\sup \{x_n | n \in \mathbb{N}\}$. Similarly, for any weakly decreasing sequence x_n (with $x_n \geq x_{n+1}$ for all n), it is natural to think that x_n converges to $\inf \{x_n | n \in \mathbb{N}\}$. The order topology is the smallest topology in which every weakly increasing sequence x_n converges to its supremum

$$\lim x_n \equiv \sup x_n,$$

and every weakly decreasing sequence x_n converges to its infimum

$$\lim x_n \equiv \inf x_n.$$

Definition 8 *A function $f : X \to Y$ (where Y is any topological space, such as $\mathbb{R}$), f is said to be* continuous (in the order topology) *if for every monotone sequence x_n, $\lim f(x_n) = f(\lim x_n)$.*

That is, for every weakly increasing sequence x_n, $\lim f(x_n) = f(\sup x_n)$, and for every weakly decreasing x_n, $\lim f(x_n) = f(\inf x_n)$.

3 Monotonicity Theorem

In this section, I will present the main result for the individual decision problems, establishing the lattice structure of the optimal solutions and establishing monotonicity of the solution to the complementary payoff parameters.

4 Monotonicity Theorem

Theorem 1 (Topkis's Monotonicity Theorem) *For any lattices* $(X, \geq)$ *and* $(T, \geq)$*, let* $f : X \times T \to \mathbb{R}$ *be a supermodular function (with coordinate-wise order) and define*

$$x^* (t) = \arg \max_{x \in S(t)} f (x, t) .$$

If $t \geq t'$ *and* $S(t) \geq S(t')$*, then* $x^*(t) \geq x^*(t')$*.*

A couple of comments on the statement of the result are in order. First, we use the strong set order in comparing $S(t)$ to $S(t')$ and $x^*(t)$ to $x^*(t')$. Using such a strong notion to compare the domains make the result weak, but its usage in comparison of the optimal solutions makes the result strong. Second, the supermodularity condition on f here can be weakened as

$$f (x \vee x', t) + f (x \wedge x', t') \geq f (x, t) + f (x', t')$$

because we are only interested in the case of t and t'. Finally, the condition that $t \geq t'$ can always be satisfied as a matter of definition.

Proof. Assuming $t \geq t'$ and $S(t) \geq S(t')$, take any $x \in x^*(t)$ and $x' \in x^*(t')$. In order to show that $x^*(t) \geq x^*(t')$, we need to show that $x \vee x' \in x^*(t)$ and $x \wedge x' \in x^*(t')$. For this, it suffices to show that $x \vee x' \in S(t)$, $x \wedge x' \in S(t')$, $f(x \vee x', t) = f(x, t)$ and $f(x \wedge x', t) = f(x', t')$. First, since $x \in x^*(t) \subseteq S(t)$, $x \in S(t)$. Similarly, $x' \in S(t')$. Since $S(t) \geq S(t')$, we then have $x \vee x' \in S(t)$ and $x \wedge x' \in S(t')$. To show $f(x \vee x', t) = f(x, t)$ and $f(x \wedge x', t') = f(x', t')$, note that since $x \in x^*(t)$ and $x \vee x' \in S(t)$, $f(x \vee x', t) \leq f(x, t)$. Similarly, $f(x \wedge x', t') \leq f(x', t')$. If either of these inequalities were strict, we would have

$$f (x \vee x', t) + f (x \wedge x', t') < f (x, t) + f (x', t') ,$$

contradicting the supermodularity condition above. Therefore, $f(x \vee x', t) = f(x, t)$ and $f(x \wedge x', t') = f(x', t')$. ■

Note that when the domain of optimization is a lattice, the Monotonicity Theorem implies that the optimal solutions form a lattice:

Corollary 2 *For any fixed t, if $f(\cdot, t) : X \to \mathbb{R}$ is supermodular and $S(t)$ is a sublattice of X, then $x^*(t)$ is a sublattice of X.*

Proof. Since $S(t)$ is a sublattice, $S(t) \geq S(t)$. Since $t \geq t$, Monotonicity Theorem concludes that $x^*(t) \geq x^*(t)$, showing that $x^*(t)$ is a sublattice. ■

Note that Monotonicity Theorem leads to strong comparative statics without making any continuity assumption or any assumption on the domain of the parameters t. For example consider the function f on Figure 2, where $T = \{0, 1\}$ and $f(x, 1) - f(y, 1) \geq f(x, 0) - f(y, 0)$ for any $x > y$. Let $S(0) = [0, 2]$ and $S(1) = [a, a+2]$ for $a \geq 0$. Considering $(t = 1, a)$ as the new parameter, note that S is increasing in both t and a. Monotonicity Theorem concludes that x^* is increasing in t and a. Indeed,

$$\begin{aligned} x^*(0) &= \{x_0\} \\ x^*(1, a) &= \begin{cases} \{a+2\} & \text{if } a+2 \leq x_1 \\ \{x_1\} & \text{if } x_1 \leq a+2 < x_2 \\ \{x_1, x_2\} & a+2 = x_2 \\ \{a+2\} & \text{otherwise.} \end{cases} \end{aligned}$$

Since $a \geq 0$, $x^*(1, a) \geq x^*(0)$. This is despite the fact that the solution is discontinuous and f does not satisfy the usual concavity conditions. This example also shows that the assumption that $S(t) \geq S(t')$ is not superfluous. If $a < x_0 - 2$, so that $S(1, a) \not\geq S(0)$, then $x^*(1, a) = \{a+2\} \not\geq \{x_0\} = x^*(0)$.

4.1 Applications

I will illustrate the applications of Monotonicity Theorem on a couple traditional examples next.

Example 4 (Pricing) *Consider a monopolist who chooses a price p for its product, facing a demand function $D(p, t)$ and marginal cost c, where t is a demand parameter.*

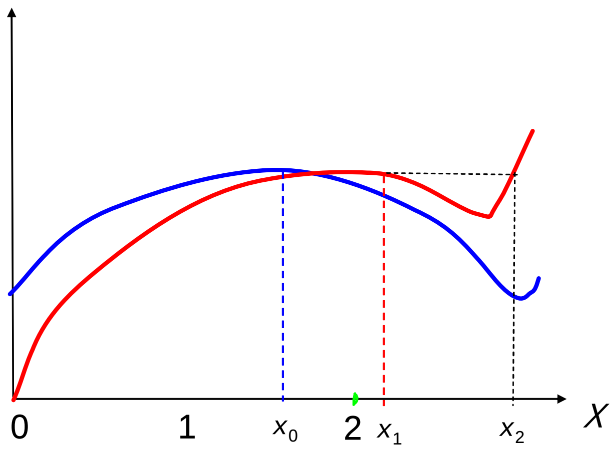

Figure 2:

Write

$$p^*(t,c) = \arg\max_{p \geq c'} (p-c)\, D(p,t)$$

for the optimal price, where $c' > c$ is a fixed lower bound for prices. Application of Monotonicity Theorem to this problem may not be as useful. Observe that optimal solution is invariant to monotone transformations of objective functions, and hence

$$p^*(t,c) = \arg\max_{p \geq c'} \log(p-c) + \log D(p,t).$$

The new objective function is supermodular with respect to p and c. Hence, Monotonicity Theorem concludes that p^* is weakly increasing in c. Moreover, the new objective function is supermodular with respect to p and c as long as $\log D(p,t)$ is supermodular (i.e. $D(p,t)$ is log-supermodular). Hence, Monotonicity Theorem concludes that p^* is weakly increasing in c as long as the demand function is log-supermodular.

Note also that the order on the domain is invariant to monotone transformations on the domain. Hence, the latter condition is equivalent to $\log D(p,t)$ being supermodular with respect to $(\log p, t)$, i.e. the price elasticity of demand

$$-\frac{\partial \log D(p,t)}{\partial \log p}$$

being weakly decreasing in t.

Example 5 (Auction Theory) *Consider a bidder in an auction for an object. The value of winning the object at price/bid p is $U(p,t)$ where t is the type of the player.*

Suppose that we are only interested in whether the bidder's bid is increasing in his type (which ensures optimality of the auctions under certain conditions). When can we ensure that the bid is indeed increasing in t without computing the solution to the entire auction problem, which is often very difficult? Write $F(p)$ for the probability of winning when he bids p. The optimal bid is

$$\begin{aligned} p^*(t) &= \arg\max U(p,t) F(p) \\ &= \arg\max \log U(p,t) + \log F(p). \end{aligned}$$

By Monotonicity Theorem, $p^*(t)$ is weakly increasing in t as long as U is log-supermodular.

Exercise 1 *Note that the above analysis assumes that the probability of winning is independent of type given the bid, which makes sense only if the types are independent. How would the answer change if F depends on both p and t?*

Example 6 (Production) *Suppose that the profit of a firm is*

$$pf(k,l) - L(l,w) - K(k,r),$$

where p is the price of the product, k is the capital input, l is the labor input, and w and r are cost parameters for labor and capital, such as wage and interest, respectively. The firm chooses k and l. How would an increase in w affect the optimal labor and capital level? Note that it is natural to assume that L is supermodular (e.g. $L = lw$). This is equivalent to assuming that $-L$ is supermodular with respect to $-l$. It turns out that this suffices to conclude that optimal l is weakly decreasing in w. Note that in order to apply Monotonicity Theorem, we need to ensure that the profit function is supermodular in (k,l,w), i.e., we also need to assume supermodularity with respect to k. We go around this requirement as follows (which is a useful technique/trick). Since we are only interested how l changes with w, the order on k is irrelevant to the end result. Let $l^*(w,r)$ and $l^*(w',r)$ be the solutions at the relevant values where $-l^*(w,r) \geq -l^*(w',r)$. We order k in such a way that the profit function is supermodular at these values:

$$k \geq k' \iff f(k, l^*(w,r)) - f(k, l^*(w',r)) \geq f(k', l^*(w,r)) - f(k', l^*(w',r)).$$

Then, Monotonicity Theorem (with the restricted domain) implies that $-l$ is weakly increasing in w, i.e., l is weakly decreasing in w.

How does optimal k change in w? In order to answer this question we need to use the original (or reverse) order on k. It is usually assumed that the production function is supermodular (e.g. $f = k^{\alpha}l^{\beta}$). In that case, the profit function is supermodular when we use the reverse order on both k and l. Then, Monotonicity Theorem concludes that the optimal capital k is decreasing in w.

4.2 Monotonicity Theorem with Continuity and Completeness

In application to supermodular games, we will assume that the strategy spaces are complete lattices and the utility functions are continuous with respect to the order topology. In that case the optimal solutions have further properties.

Theorem 2 *Let $(X, \geq)$ be a lattice, and $f : X \to \mathbb{R}$ be supermodular and continuous in the order topology. Then, for any complete sublattice S,*

$$x^* = \arg\max_{x \in S} f(x)$$

is a complete sublattice, and $\max x^* \in x^*$ and $\min x^* \in x^*$ exist.

Proof. The fact that x^* is a lattice follows from Topkis's Monotonicity Theorem as we have seen above. Moreover, for every $A \subseteq x^* \subseteq S$, $\inf A \in S$ and $\sup A \in S$ exist because S is complete. It suffices to show that $\inf A \in x^*$ and $\sup A \in x^*$. This is immediate because we can construct a weakly decreasing sequence x_n in A with $\inf x_n = \inf A$ and the continuity of f implies that $f(\inf A) = f(\inf x_n) = \lim f(x_n) = \lim f(x_0)$ because f is constant over x^*. Since x_n is a solution and $\inf A \in S$, this implies that $\inf A \in x^*$. The argument for $\sup A \in x^*$ is identical. Since $\inf x^* \in x^*$ and $\sup x^* \in x^*$, we write $\min x^* = \inf x^*$ and $\max x^* = \sup x^*$. ■

5 Supermodular Games

We will now consider games in which the strategy spaces are complete lattices and the utility functions are continuous (with respect to the order topology) and supermodular (we will make a slightly weaker assumption). Such games are called supermodular. For these games we will establish a useful structure of Nash equilibria and rationalizability, showing that the rationalizable strategies are bounded by extremal equilibria, and obtain a useful monotonicity result on extremal equilibria. We will conclude by introducing incomplete information to the analysis.

5.1 Formulation

Definition 9 *A game (N, S, u) is supermodular if for each player $i \in N$,*

- *strategy space $(X_i, \geq_i)$ is a complete lattice for some order $\geq_i$, and*
- *u_i is continuous with respect to the product topology, supermodular in x_i and has increasing differences:*

$$u_i(x \vee x) + u_i(x \wedge x') \geq u_i(x) + u_i(x') \qquad \left(\forall x_i, x'_i \in X_i, \forall x_{-i} \geq x'_{-i} \in X_{-i}\right).$$

Since X is a complete lattice $\underline{x} = \min X$ and $\bar{x} = \max X$ exist.

Here, recall that continuity of u_i means that for every weakly increasing sequence of strategy profiles $x(n)$, $\lim_n u_i(x(n)) = u_i(\sup_n x(n))$, and for every weakly decreasing sequence of strategy profiles $x(n)$, $\lim_n u_i(x(n)) = u_i(\inf_n x(n))$.

The supermodularity condition is weaker than full supermodularity because we only consider strategy profiles x_{-i} and x'_{-i} for the other players that are ordered. When we restrict $x_{-i} = x'_{-i}$, the above condition reduces to the condition that u_i is supermodular in x_i:

$$u_i(x_i \vee x'_i, x_{-i}) + u_i(x_i \wedge x'_i, x_{-i}) \geq u_i(x_i, x_{-i}) + u_i(x'_i, x_{-i}) \qquad (\forall x_i, x'_i \in X_i, \forall x_{-i} \in X_{-i}).$$

When we restrict x_i and x'_i to be ordered, say $x'_i \geq x_i$, the above condition reduces to the usual increasing differences condition:

$$u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i}) \geq u_i\left(x'_i, x'_{-i}\right) - u_i\left(x_i, x'_{-i}\right) \qquad \left(\forall x'_i \geq x_i \in X_i, \forall x_{-i} \geq x'_{-i} \in X_{-i}\right).$$

These are the only restrictions imposed by the definition. Recall that in product lattices as in here, supermodularity is equivalent to supermodularity with respect to each x_j and increasing differences. Here, we assume supermodularity with respect to x_i and increasing differences, but we do not make any supermodularity assumption with respect to other players' strategies x_j (with $j \neq i$). This is the only weakening of supermodularity.

Example 7 (Linear Differentiated Bertrand Oligopoly) *Consider a Bertrand oligopoly model with each player i faces constant marginal cost c_i and demand function $Q_i(p) = A - a_i p_i + \sum_{j \neq i} b_j p_j$, where A, a_i and b_j are all positive numbers. Restrict the choice of possible price p_i for each i to be in $[c_i, \bar{p}_i]$ for some large $\bar{p}_i$. This yields a supermodular game in the natural order because*

$$u_i(p) = (p_i - c_i) Q_i(p)$$

is supermodular:

$$\frac{\partial^2 u_i}{\partial p_i \partial p_j} = b_j \geq 0.$$

Example 8 (Linear Cournot Duopoly) *Consider a Cournot duopoly model with inverse demand function $P = A - q_1 - q_2$ and cost functions $C_1(q_1)$ and $C_2(q_2)$. Restrict the set of possible production levels to a large compact interval. This leads to a "submodular" game in the natural order because the utility function of firm i is*

$$u_i(q) = q_i P(q) - C_i(q_i),$$

yielding

$$\frac{\partial^2 u_i}{\partial q_i \partial q_j} = -1 < 0.$$

This is a supermodular game when q_2 is ordered in the reverse order:

$$\frac{\partial^2 u_i}{\partial q_1 \partial(-q_2)} = 1 > 0.$$

In general, submodular two player games can be made supermodular by reversing the order on one of the strategies. Hence, submodular two player games exhibit the useful properties of supermodular games. This trick does not work, however, when there are more than two players, and the submodular games with more than two players may exhibit dramatically different properties than the supermodular ones.

Example 9 (Linear Cournot Oligopoly) *In the above example, suppose that there are three or more players. Once again, for any $i \neq j$,*

$$\frac{\partial^2 u_i}{\partial q_i \partial q_j} = -1 < 0.$$

But this game cannot be made supermodular by reversing the orders. Indeed, the relation between rationalizability and Nash equilibria in Cournot oligopoly is quite different than the relation in Cournot duopoly, as we will see later.

5.2 Rationalizability and Equilibrium

In this section, we will establish that (i) there exist extremal equilibria and that (ii) all rationalizable strategies are bounded by the extremal equilibria. We start with summarizing the useful implications of the monotonicity results in previous section and introduce a couple useful notation.

Lemma 2 *For any supermodular game, any $i \in N$,*

1. *for every $x_{-i} \in X_{-i}$, $x_i^* (x_{-i}) = \arg\max_{x_i \in X_i} u_i (x_i, x_{-i})$ is a complete lattice;*
2. *for every x, $B_i (x) \equiv \max x_i^* (x_{-i}) \in x_i^* (x_{-i})$ and $b_i (x) \equiv \min x_i^* (x_{-i}) \in x_i^* (x_{-i})$, and*
3. *B_i and b_i are isotone, i.e., $B_i (x) \geq B_i (y)$ and $b_i (x) \geq b_i (y)$ whenever $x \geq y$.*

Proof. The first two statements are by the monotonicity theorem for complete lattices (namely Theorem 2). But since u_i is supermodular with increasing differences and the domain of optimization is independent of x_{-i}, by Topkis's Monotonicity Theorem, whenever $x \geq y$, $x_i^* (x_{-i}) \geq x_i^* (y_{-i})$ in the sense of strong set order. In particular, $B_i (x) = \max x_i^* (x_{-i}) \geq \max x_i^* (y_{-i}) = B_i (y)$ and $b_i (x) = \min x_i^* (x_{-i}) \geq \min x_i^* (y_{-i}) = b_i (y)$. ■

The following lemma will be the main step in establishing the extremal rationalizable strategies and equilibria.

Lemma 3 *Every x_i with $x_i \not\geq b_i (\underline{x})$ is strictly dominated by $x_i \vee b_i (\underline{x})$, where $\underline{x} = \min X$.*

Proof. Take any x_{-i}. We want to show that

$$u_i (x_i \vee b_i (\underline{x}), x_{-i}) - u_i (x_i, x_{-i}) > 0. \tag{1}$$

Now, since $x_{-i} \geq \underline{x}_{-i}$ and $x_i \vee b_i (\underline{x}) \geq x_i$,we have

$$\begin{aligned} u_i (x_i \vee b_i (\underline{x}), x_{-i}) - u_i (x_i, x_{-i}) &\geq u_i \left(x_i \vee b_i (\underline{x}), \underline{x}_{-i}\right) - u_i \left(x_i, \underline{x}_{-i}\right) \\ &\geq u_i \left(b_i (\underline{x}), \underline{x}_{-i}\right) - u_i \left(x_i \wedge b_i (\underline{x}), \underline{x}_{-i}\right). \end{aligned} \tag{2}$$

where the first inequality is by increasing differences and the second inequality supermodularity in own strategy x_i. Hence, to show (**??**), it suffices to show that

$$u_i \left(b_i (\underline{x}), \underline{x}_{-i}\right) - u_i \left(x_i \wedge b_i (\underline{x}), \underline{x}_{-i}\right) > 0.$$

But, since $b_i(\underline{x}) \in \arg\max_{x_i} u_i\left(x_i, \underline{x}_{-i}\right)$, $u_i\left(b_i(\underline{x}), \underline{x}_{-i}\right) \geq u_i\left(x_i \wedge b_i(\underline{x}), \underline{x}_{-i}\right)$. If it were true that $u_i\left(b_i(\underline{x}), \underline{x}_{-i}\right) = u_i\left(x_i \wedge b_i(\underline{x}), \underline{x}_{-i}\right)$, then we would have $x_i \wedge b_i(\underline{x}) \in \arg\max_{x_i} u_i\left(x_i, \underline{x}_{-i}\right)$, and by definition of $b_i(\underline{x})$ we would have $x_i \wedge b_i(\underline{x}) \geq b_i(\underline{x})$, showing that $x_i \geq b_i(\underline{x})$, a contradiction. Therefore, $u_i\left(b_i(\underline{x}), \underline{x}_{-i}\right) > u_i\left(x_i \wedge b_i(\underline{x}), \underline{x}_{-i}\right)$. ■

Iterative application of this lemma leads to the following well-known result, due to Milgrom and Roberts.

Theorem 3 *For any supermodular game,*

1. $\bar{z} \equiv \lim_k B^k(\bar{x}) \equiv \inf_k B^k(\bar{x})$ *and* $\underline{z} \equiv \lim_k b^k(\underline{x}) \equiv \sup_k b^k(\underline{x})$ *exists, where* $\bar{x} = \sup X$ *and* $\underline{x} = \inf X$*;*

2. *for every rationalizable strategy profile* x*,*

$$\bar{z} \geq x \geq \underline{z},$$

3. *and* $\bar{z}$ *and* $\underline{z}$ *are (pure strategy) Nash equilibria.*

Proof. (Part 1) First note that $B^k(\bar{x})$ is weakly decreasing. $[B^1(\bar{x}) \leq B^0(\bar{x}) = \bar{x}$ by definition of $\bar{x}$. If $B^k(\bar{x}) \leq B^{k-1}(\bar{x})$, then by monotonicity of B (Lemma 2), $B^{k+1}(\bar{x}) = B\left(B^k(\bar{x})\right) \leq B\left(B^{k-1}(\bar{x})\right) = B^k(\bar{x})$.] Hence, $\lim_k B^k(\bar{x}) = \inf_k B^k(\bar{x})$ (existence is by completeness, of course, as we have seen before). Similarly, $b^k(\underline{x})$ is weakly increasing, and therefore $\lim_k b^k(\underline{x}) \equiv \sup_k b^k(\underline{x})$.

(Part 2) I will show that if $x_i \in S_i^k$, then $x_i \geq b_i^k(\underline{x})$. This is true for $k = 0$, by definition. Suppose that $x_j \geq b_j^{k-1}(\underline{x})$ for all $j \in N$ and $x_j \in S_j^k$. Then, by Lemma 3, every $x_i \not\geq b_i^{k-1}(\underline{x})$ is strictly dominated in the reduced game at round k and is not in S_i^k. Therefore, $x_i \geq b_i^k(\underline{x})$ for every $x_i \in S_i^k$.

(Part 3) I will show that $\bar{z}$ is a Nash equilibrium, i.e., $\bar{z}_i \in x_i^*(\bar{z}_{-i})$. To this end, take any x_i, and consider the weakly decreasing sequences $\left(x_i, b_{-i}^{k-1}(\bar{x})\right)$ and $\left(b_i^k(\bar{x}), b_{-i}^{k-1}(\bar{x})\right)$. Clearly, $\lim_k \left(x_i, b_{-i}^{k-1}(\bar{x})\right) \to (x_i, \bar{z}_{-i})$ and $(\bar{z}_i, \bar{z}_{-i})$. Moreover, since $b_i^k(\bar{x}) \in x^*\left(b_{-i}^{k-1}(\bar{x})\right)$, $u_i\left(b_i^k(\bar{x}), b_{-i}^{k-1}(\bar{x})\right) \geq u_i\left(x_i, b_{-i}^{k-1}(\bar{x})\right)$ for each k. Hence, by continuity of u_i in the order topology,

$$\begin{aligned} u_i(\bar{z}_i, \bar{z}_{-i}) &= u_i\left(\lim\left(b_i^k(\bar{x}), b_{-i}^{k-1}(\bar{x})\right)\right) = \lim u_i\left(b_i^k(\bar{x}), b_{-i}^{k-1}(\bar{x})\right) \\ &\geq \lim u_i\left(x_i, b_{-i}^{k-1}(\bar{x})\right) = u_i\left(\lim\left(x_i, b_{-i}^{k-1}(\bar{x})\right)\right) = u_i(x_i, \bar{z}_{-i}). \end{aligned}$$

■

This result establishes several important fact. First of all, it establishes that there exists an equilibrium, indeed, extremal equilibria in pure strategies (Part 3). Second, it establishes a useful procedure to compute these equilibria (Part 3), namely, one iteratively applies extremal best response functions to the largest and smallest strategy profiles. In comparison, finding a fixed point of a function is a computationally hard problem. Finally, it establishes that the rationalizable strategies are bounded by these extremal equilibria (Part 3). This not only relates extreme implications of equilibrium and rationalizability to each other, but also helps in identifying rationalizable strategies. For example, when the extremal best response functions are continuous and strategy sets are convex intervals, the result implies that the rationalizable set is the convex hull of extremal equilibrium strategies. It also implies that uniqueness of rationalizability is equivalent to dominance solvability:

Corollary 3 *A supermodular game is dominance solvable if and only if there exists a unique Nash equilibrium in pure strategies.*

The following example illustrates the Milgrom-Roberts theorem above and shows that completeness is not superfluous.

Example 10 (Partnership Game) *There is an employer, who provides capital K, and a worker, who provides labor L. They share the output, which is $K^\alpha L^\beta$ for some $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$. The utility functions of the Employer and the Worker are $K^\alpha L^\beta / 2 - K$ and $K^\alpha L^\beta / 2 - L$, respectively. The best-response functions K^* and L^* are plotted in Figure 3. There are two pure strategy equilibria, one at $(0,0)$ and one with positive labor and capital, denoted by $\left(\hat{K}, \hat{L}\right)$. When all nonnegative inputs are allowed, the strategy sets are not complete lattices. In that case, every strategy is a best response to some other, and hence every strategy is rationalizable, and the bounds of Milgrom and Roberts are not valid. Now suppose that the strategy sets are bounded by above for some large $\bar{K}$ and $\bar{L}$, so that $K \in \left[0, \bar{K}\right]$ and $L \in \left[0, \bar{L}\right]$. Now, we have a supermodular game (with complete lattices as strategy spaces). Then, as shown in the figure, one can iteratively eliminate all $K > \hat{K}$ and $L > \hat{L}$. Hence $S^\infty \subseteq \left[0, \hat{K}\right] \times \left[0, \hat{L}\right]$, as in the Milgrom-Roberts theorem. Moreover, since the best response functions are continuous, $\left[0, \hat{K}\right] \times \left[0, \hat{L}\right]$ is closed under rational behavior, and hence $S^\infty = \left[0, \hat{K}\right] \times \left[0, \hat{L}\right]$.*

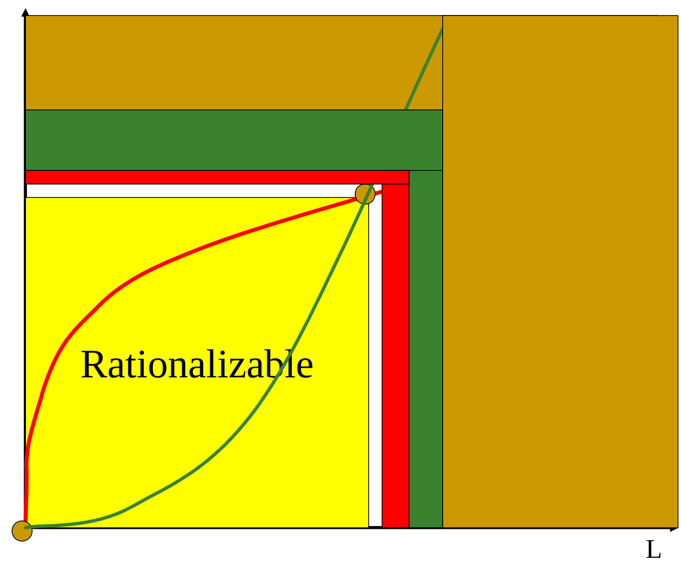

Figure 3: Rationalizability and Equilibria in the partnership game

5.3 Comparative Statics

The next result, due to Milgrom and Roberts, shows that the extremal equilibria are weakly increasing in complementary parameters, extending the Monotonicity Theorem for optimization in games.

Theorem 4 *Consider a family of supermodular games with payoffs parameterized by t. Suppose that for all i, x_{-i}, $U_i(x_i, x_{-i}; t)$ is supermodular in (x_i, t). Write $\bar{z}(t)$ and $\underline{z}(t)$ for the extremal equilibria at t. Then, $\bar{z}(t)$ and $\underline{z}(t)$ are isotone.*

Proof. Take any t, t' with $t \geq t'$, and write b_t and $b_{t'}$ for the minimal best response function under t and t', respectively. By Topkis's Monotonicity Theorem, $b_t(x) \geq b_{t'}(x)$ for every x. Since b_t and $b_{t'}$ are isotone, this further implies that $b_t^k(x) \geq b_{t'}^k(x)$ for every k. Therefore,

$$\underline{z}(t) = \sup b_t^k(\underline{x}) \geq \sup b_{t'}^k(\underline{x}) = \underline{z}(t').$$

Similarly,

$$\bar{z}(t) = \inf B_t^k(\bar{x}) \geq \inf B_{t'}^k(\bar{x}) = \bar{z}(t').$$

■

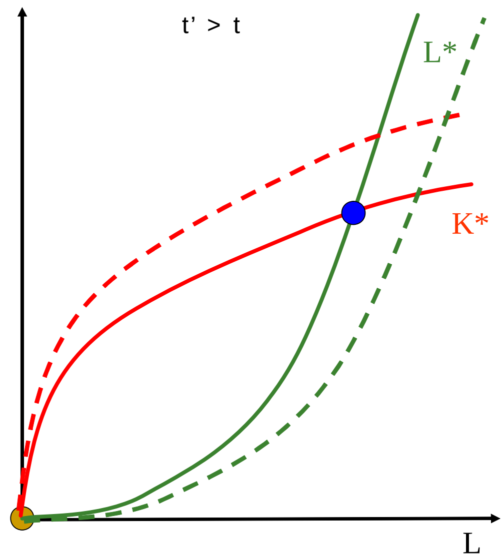

Figure 4: Effect of productivity parameter in the partnership game

Example 11 *As an illustration of the theorem, take the output function in the partnership game as* $tK^{\alpha}L^{\beta}$*. As it is illustrated in Figure 4, an increase in* t *results in steeper best response functions. This leads the largest equilibrium to increase. On the other hand, the smallest equilibrium remains unchanged (corresponding to a weak increase).*

Topkis's Monotonicity Theorem established that in single-person decision problems, the entire set of the solutions increase in the sense of strong set order. It is tempting to conjecture that the same is true for Nash equilibria in multi-person decision problems (as in the partnership game above). This is not true in general. Indeed, in Diamond's search model at the beginning, although the extremal equilibria weakly increase, the middle equilibrium actually decreases, as shown in Figure 1.

Note that S^{∞} weakly increases in the sense of set order all of these examples. Indeed, when the best response functions are continuous and strategy spaces are one-dimensional, $S_i^{\infty} = [\underline{z}_i, \bar{z}_i]$, and the monotonicity result of Milgrom and Roberts already imply that S^{∞} is isotone in the sense of strong set order. It is not clear how general this fact is.

6 Supermodular Bayesian Games

To some extend, the analyses for complete information games above contains Bayesian games with countable type spaces because any such Bayesian game can be represented by the interim game as a game of complete information. After illustrating this fact, I will introduce monotone supermodular games of incomplete information (due to Vives and van Zandt), in which the type and action spaces are compact subsets of $\mathbb{R}^n$, and the players' beliefs are monotone with respect to their types. These games will then exhibit further monotonicity properties because of belief monotonicity. This analysis will be used later in global games, which are special cases of these games.

I will start with illustrating how one can use the above results for Bayesian games with countable (or finite) type spaces.

Definition 10 *A countable supermodular Bayesian game is a Bayesian game* $\mathcal{B} = (N, A, \Theta, T, u, p)$ *with*

- *each* A_i *is a complete lattice for some* $\geq_i$*,*
- T *is countable (or finite), and*
- u_i *is measurable, bounded, continuous in* a*, supermodular in* a_i *and has increasing differences.*

As in the earlier lectures, for any such Bayesian game, one can define the interim game $AG(\mathcal{B})$, by taking $\cup_i T_i$ as the countable set of players, A_i as the action space for each t_i, and

$$u_{t_i}(s) = E\left[u_i\left(\theta, s_i\left(t_i\right), s_{-i}\left(t_{-i}\right)\right) | t_i\right],$$

where s is taken as a profile of actions (for all types), rather than strategies in the ex-ante sense. Since u_i is supermodular in a_i, $u_{t_i}(s)$ is supermodular in $s_i(t_i)$. Since u_i has increasing differences in a_{-i}, u_{t_i} has increasing differences with respect to all actions other than $s_i(t_i)$ (type t_i puts zero probability on other types of i). Moreover, since u_i is bounded, continuity of u_i implies continuity of u_{t_i}. Since A_i is already a complete lattice, this shows that the interim game is supermodular.

Lemma 4 *For any countable supermodular Bayesian game* $\mathcal{B}$*, the interim game* $AG(\mathcal{B})$ *is a supermodular game (of complete information).*

Using this observation, one extends the previous results to Bayesian games as follows.

Theorem 5 *For any countable supermodular Bayesian game $\mathcal{B}$, the following are true.*

1. *There exist Bayesian Nash equilibria $\underline{s}$ and $\bar{s}$ in pure strategies.*

2. *For any interim independent rationalizable action a_i of any type t_i,*

$$\bar{s}_i(t_i) \geq a_i \geq \underline{s}_i(t_i).$$

3. *For any Bayesian Nash equilibrium s, $\bar{s}(t) \geq s(t) \geq \underline{s}(t)$ for all $t \in T$.*

Moreover, for any family of countable supermodular Bayesian games $\mathcal{B}^\lambda = (N, A, \Theta, T, u^\lambda, p)$ with $u_i^\lambda(\theta, a_i, a_{-i})$ supermodular in (a_i, λ), the extremal equilibria $\underline{s}^\lambda$ and $\bar{s}^\lambda$ are isotone in λ.

Proof. By Lemma 4, $AG(\mathcal{B})$ is a supermodular game. (1) Hence by Theorem 3, $AG(\mathcal{B})$ has Nash equilibria $\underline{s}$ and $\bar{s}$ in pure strategies. Of course, $\underline{s}$ and $\bar{s}$ are Bayesian Nash equilibria of $\mathcal{B}$. (2) Any interim independent rationalizable action a_i of any type t_i is a rationalizable action of t_i in $AG(\mathcal{B})$ by definition. Hence, by Theorem 3, $\bar{s}_i(t_i) \geq a_i \geq \underline{s}_i(t_i)$. Part 3 follows from Part 2. For the last statement, observe that $u_{t_i}^\lambda(s) = E\left[u_i^\lambda(\theta, s_i(t_i), s_{-i}(t_{-i})) | t_i\right]$ is supermodular in $(s_i(t_i), \lambda)$. Hence, by Theorem 4, $\underline{s}^\lambda$ and $\bar{s}^\lambda$ are isotone in λ. ■

Unfortunately, the above transformation cannot be applied to countable type spaces because one needs measurability condition on strategies in order to compute the expectation. (Hence, the space of strategy profiles is not a product set in $AG(\mathcal{B})$.) For such cases, Vives and Van Zandt introduce following class of Bayesian games, which also incorporate useful monotonicity conditions on beliefs.

Definition 11 *A monotone supermodular game (of incomplete information) is a Bayesian game $\mathcal{B} = (N, A, \Theta, T, u, p)$ with*

- *each A_i is a compact sublattice of $\mathbb{R}^K$;*
- *$\Theta \times T$ is a measurable subset of $\mathbb{R}^M$;*
- *u_i is such that*

– *$u_i(a, \cdot) : \Theta \to \mathbb{R}$ is measurable, and*

– *$u_i(\cdot, \theta) : A \to \mathbb{R}$ is continuous, bounded by an integrable function, supermodular in a_i and has increasing differences,*

– *u_i has increasing differences in (a_i, θ)*

- *$p(\cdot|t_i)$ is a weakly increasing function of t_i in the sense of 1st-order stochastic dominance.*

This definition is more general in that type spaces can be any subspace of a $\mathbb{R}^n$, but it is more restrictive in that it restricts the action spaces to be subsets of $\mathbb{R}^n$.Clearly, the continuity and measurability assumptions on u is made in order to ensure the necessary continuity of conditional expected payoffs of types. Finally, the beliefs of types are assumed to be monotone in the sense of first-order stochastic dominance. Together with supermodularity of u, this ensures that the extremal equilibria are monotone (for the same reason behind Theorem 4). This leads to the following result.

Theorem 6 *A monotone supermodular game there exists Bayesian Nash equilibria $\underline{s}$ and $\bar{s}$ in pure strategies such that for any Bayesian Nash equilibrium s, $\bar{s}(t) \geq s(t) \geq \underline{s}(t)$ for all $t \in T$, and $\bar{s}_i(t_i)$ and $\underline{s}_i(t_i)$ are weakly increasing in t_i.*

This result is silent about rationalizable strategies, but as we will see in the context of global games they are also bounded by the extremal equilibria as in the previous results.

In conclusion, in supermodular games, all rationalizable strategies are bounded by extremal pure strategy equilibria, and these equilibria are weakly increasing with respect to complementary variables, leading to monotone comparative statics.

7 Exercises

1. For some lattice $(X, \geq)$, consider supermodular functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. Prove or disprove the following.

   (a) For any $a, b \geq 0$, $af + by$ is supermodular.

(b) If f and g are isotone and nonnegative (i.e. $f(x) \geq 0$ and $g(x) \geq 0$ for all x), then fg is supermodular.

(c) Under the conditions in part b, f^{α} is supermodular for any $\alpha \geq 1$.

2. This question asks you to prove some basic facts.

(a) For any lattice $(X, \geq)$, show that if S and T are sublattices, so is $S \cap T$.

(b) Consider a function $f : X \times \Omega \to \mathbb{R}$, where $(X, \geq)$ is a lattice and Ω is a probability space with expectation operator E. Show that if $f(\cdot, \omega)$ is supermodular for all $\omega \in \Omega$, then $E[f]$ is also supermodular.

(c) Let X be a set of sets that is closed under union (i.e. $A \cup B \in X$ for all $A, B \in X$) and with $\emptyset \in X$. Define order $\geq$ on X by $A \geq B \iff A \supseteq B$. Show that $(X, \geq)$ is a complete lattice. What are $A \vee B$ and $A \wedge B$?

(d) On the set $\mathbb{R}_+^{\mathbb{R}_+}$ of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$, define order $\geq$ by $f \geq g \iff f(x) \geq g(x)$ for all $x \in \mathbb{R}$. Showthat $\left(\mathbb{R}_+^{\mathbb{R}_+}, \geq\right)$´is a lattice. Show also that the following are sublattices:

i. all continuous functions,

ii. all non-increasing functions, and

iii. all functions f with $f \leq g$ for some fixed function g.

3. Prove the following statements.

(a) If f and g are supermodular, so is $f + g$.

(b) If f is supermodular and $a > 0$, then af is also supermodular.

(c) If $f : \Theta \times X \to \mathbb{R}$, where

- X is a lattice,
- $\theta \in \Theta$ is not known,
- $f(\theta, \cdot) : X \to \mathbb{R}$ is supermodular for each $\theta \in \Theta$,

then $E[f] : X \to \mathbb{R}$ is supermodular, where E is an expectation operator on Θ.

4. A lattice X is said to be complete iff every $Y \subset X$ has a greatest lower bound and a least upper bound in X. Let X be a complete lattice and $f : X \to X$ be isotone. (Do not assume that B is continuous.) Define

$$\begin{aligned} \underline{x} &= \inf \{x | f(x) \leq x\} \\ \bar{x} &= \sup \{x | f(x) \geq x\}. \end{aligned}$$

Show that $\bar{x}$ and $\underline{x}$ are fixed points of f, i.e., $\bar{x} = f(\bar{x})$ and $\underline{x}= f(\underline{x})$. Show also that, if $f(x) = x$, then $\underline{x} \leq x \leq \bar{x}$.

5. Let $\underline{z}= (\underline{z}_1, \ldots, \underline{z}_n)$ be the smallest rationalizable strategy profile in a given supermodular game. Let also $\underline{y}$ be the smallest Nash equilibrium of the game that is created by fixing Player 1's strategy at $\underline{z}_1$. Show that

$$\underline{z} = \underline{y}.$$

6. Let X be a complete lattice and $T = \mathbb{R}$.

   (a) Let $f : X \times T \to X$ be isotone, and $\bar{x}(t)$ be the highest fixed point of $f(\cdot, t)$ for each t. Show that $\bar{x}$ is isotone. [Hint: First show that $\bar{x}(t) = \sup \{x | f(x,t) \geq x\}$.]

   (b) Let $B_n(x,t)$ be the largest best reply to x_{-n} for each n in a supermodular game G_t with a generic strategy profile $x = (x_1, \ldots, x_N)$. Let also $B(x,t) = (B_1(x,t), \ldots, B_N(x,t))$. Let $\bar{x}(t)$ be the highest Nash equilibrium of G_t. Show that, if $t \geq t'$, then

$$\bar{x}(t) \geq B(\bar{x}(t'), t).$$

7. There is a (large) consumer of a good with integrable, non-increasing demand function $D_t : \mathbb{R}_+ \to \mathbb{R}_+$ where $t \in \mathbb{R}$ is a demand parameter in which $D_t(q)$ is increasing for each quantity q. Consumer faces an increasing, continuous supply function $S_\omega : \mathbb{R}_+ \to \mathbb{R}_+$ where ω is an unknown supply parameter (i.e. supply is stochastic). Consumer submits a non-increasing, continuous demand function (or bid) $x : \mathbb{R}_+ \to \mathbb{R}_+$with $x \leq D_t$, and buys quantity $q(x, S_\omega)$ at price $p(x, S_\omega)$ where

$p(x, S_\omega) = x(q(x, S_\omega)) = S_\omega(q(x, S_\omega))$ (i.e. market clearing price and quantity). The payoff of consumer is

$$u(x, \omega, t) = \int_0^{q(s,S_\omega)} D_t(q)\, dq - p(x, S_\omega)q(x, S_\omega).$$

His expected utility is $U(x, t) = E[u(x, \omega, t)]$.

(a) Show that U is supermodular with respect to the order in problem 2.d.

(b) For any t, show that $x^*(t)$ is a sublattice where

$$x^*(t) = \arg\max_x U(x, t).$$

(c) Show that $x^*(t)$ is isotone in t.

8. Consider a Cournot duopoly where each firm i has a privately known cost function $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ and the inverse-demand function P is twice differentiable with $P'' + P' < 0$. Putting the order in problem 2.d on functions, assume that the set of cost functions and strategies are restricted in such a way that the strategy space is a complete lattice.

   (a) Show that there exist Nash equilibria $(\bar{x}_1, \underline{x}_2)$ and $(\underline{x}_1, \bar{x}_2)$ such that for each ex-ante rationalizable strategy x_i of each firm i, $\underline{x}_i \leq x_i \leq \bar{x}_i$.

   (b) Suppose that we add a constant $\Delta > 0$ to the inverse demand, so that the new price is $\tilde{P}(q) = P(q) + \Delta$ for each q. Can you use Milgrom-Roberts theorem to determine how $\underline{x}_i$ and $\bar{x}_i$. change?

   (c) Suppose that Firm 1 receives a government subsidy, receiving $s > 0$ for each unit it sells. Show that $\underline{x}_1$ and $\bar{x}_1$ are weakly increasing in s and $\underline{x}_2$ and $\bar{x}_2$ are weakly decreasing in s.

9. We have a differentiated Bertrand duopoly in which each firm sells m goods, $k = 1, 2, \ldots, m$. Firms 1 and 2 simultaneously set price vectors p_1 and p_2 and firm i gets profit

$$U_i = \sum_{k=1}^{m} (p_{i,k} - c_{i,k})\, Q_{i,k}(p_i, p_j)$$

where $c_{i,k} \in [0,1]$ is the constant marginal cost of producing good k for i and $Q_{i,k}$ is the demand for good k of i; it is continuous, decreasing in $p_{i,k}$, increasing in all the other variables (i.e. all $p_{j,k'}$ with $(j,k') \neq (i,k)$) and supermodular. Assume that it must be that $p_{i,k} \in [c_{i,k}, 1]$ for all i, k.

(a) Assuming that each $c_{i,k}$ is common knowledge, show that this game is supermodular. (State any additional assumption needed.)

(b) In part (a), show that there exist Nash equilibria (p_1^*, p_2^*) and (p_1^{**}, p_2^{**}) such that $p_{i,k}^* \geq p_{i,k} \geq p_{i,k}^{**}$ for each rationalizable strategy p_i and each (i,k).

(c) Assume that each $c_i = (c_{i,1}, \ldots, c_{i,m})$ is private information of i, coming from a countable subset of $[0,1]$. Show that there exist Bayesian Nash equilibria (p_1^*, p_2^*) and (p_1^{**}, p_2^{**}) such that $p_{i,k}^*(c_i) \geq p_{i,k}(c_i) \geq p_{i,k}^{**}(c_i)$ for each rationalizable strategy p_i and each (i,k,c_i).

10. Consider a partnership game with two players, who invest in a public good project at each date $t \in T = \{0,1,2,\ldots\}$ without observing each other's previous investments. We assume that a strategy of a player i is any function $x_i : T \to [0,1]$, where $x_i(t)$ is the investment level of i at $t \in T$. The payoff of of a player i is

$$U_i(x_1, x_2) = \sum_{t \in T} \delta^t \left[Af(x_1(t), x_2(t)) - c_i(x_i(t), t) \right]$$

where $\delta \in (0,1)$, $A \in [0,1]$ is a productivity parameter, $f : [0,1]^2 \to \mathbb{R}$ is a supermodular, increasing, and continuous production function, and c_i is a time dependent cost function for player i. Everything is common knowledge.

(a) Show that the above game have equilibria $\underline{x}$ and $\bar{x}$ such that for each equilibrium x of this game,

$$\underline{x}_i(t) \leq x_i(t) \leq \bar{x}_i(t) \qquad (\forall i, t).$$

(b) Let X be the set of all equilibria of this game. Construct an incomplete-information model in which (i) it is common knowledge that each player is rational and (ii) a strategy profile x is played at some state ω if and only if $x \in X$.

(c) Show that, if $A \geq A'$, then the extremal equilibria for these parameters satisfy

$$\underline{x}_i(t; A) \geq \underline{x}_i(t; A') \text{ and } \bar{x}_i(t; A) \geq \bar{x}_i(t; A') \quad (\forall i, t).$$

(d) Consider a strategy x_i with $x_i(0) > \bar{x}_i(0)$. Can you construct an incomplete-information model such that (i) each player is rational at each state and (ii) x_i is played by player i at some state?

11. Consider a two-person partnership game. Simultaneously, each player i invests $a_i \in [0,1]$, and the payoff of player i is

$$u_i(a_1, a_2, \theta) = \theta f(a_1) f(a_2) - c(a_i),$$

where $\theta \geq 0$ is a parameter, and f and c are strictly increasing functions with $f(0) > 0$. Assume that θ is common knowledge.

(a) Show that the game is supermodular.

(b) Assuming that best-reply correspondence is convex-valued and continuous, compute all rationalizable strategies. How would your answer change without the continuity assumption?

(c) Show that the minimum and the maximum rationalizable strategies as well as minimum and maximum equilibrium strategies are increasing functions of θ. Give an example, showing that set of Nash equilibria is not increasing in θ in the sense of strong set order.

12. (This question is to illustrate how we can use the ideas in supermodular game literature for structural estimation, where computational costs are very high.) Two discount chains, Walmart and Kmart, are competing for M (geographical) markets. For each market m, decision of a chain i is binary: $D_{i,m} = 1$ if it has a store in market m, and $D_{i,m} = 0$ if it does not have a store in market m. Simultaneously, each chain decides in which markets it will have a store. The profit of chain i is

$$\Pi_i = \sum_{m \in M} \left[D_{i,m} \left(\beta_i X_m + \delta_{i,j} D_{j,m} \right) + \delta_{i,i} \sum_{l \in N_m} D_{i,l} \right]$$

where

- β_i is a chain specific constant,
- X_m is a market size variable,
- $\delta_{i,j} < 0$ is a parameter measuring the competition between the two firms,
- $\delta_{i,i} \geq 0$ is a constant measuring the synergy between the neighboring stores, and
- N_m is the set of neighboring markets of market m.

Everything is common knowledge.

(a) Let

$$B_i(D_j) = \arg\max_{D_i} \Pi_i(D_i, D_j).$$

Show that computing $B_i(D_j)$ by brute force requires at least 2^M utility comparisons. How large is this number if $M = 2065$ (the number of markets in the US)? How many utility comparisons that we would have to make if we want to compute the pure strategy Nash equilibria by brute force? Comment on how long it would take an econometrician to estimate $(\beta_i, \delta_{i,i}, \delta_{i,j})$ this way?

(b) Let $F_i(D_j)$ be the set of D_i that satisfy the first-order conditions in computing $B_i(D_j)$, i.e., $D_i \in F_i(D_j)$ iff

$$\begin{aligned} D_{i,m} &= 1 \Rightarrow \Pi_i(D_i, D_j) \geq \Pi_i(0, D_{i,-m}, D_j), \\ D_{i,m} &= 0 \Rightarrow \Pi_i(D_i, D_j) \geq \Pi_i(1, D_{i,-m}, D_j), \end{aligned}$$

where $D_i' = (A, D_{i,-m})$ is the decision where $D_{i,m}' = A$ and $D_{i,m'}' = D_{i,m'}$ for all $m' \neq m$.

i. Show that $\bar{F}_i(D_j) = \max F_i(D_j)$ and $\underline{F}_i(D_j) = \min F_i(D_j)$ exists.

ii. Show that $B_i(D_j) \subseteq F_i(D_j)$, and for each $D_i \in B_i(D_j)$, $\bar{F}_i(D_j) \geq D_i \geq \underline{F}_i(D_j)$.

iii. Using the techniques discussed in the class, find a procedure for computing $\bar{F}_i(D_j)$ and $\underline{F}_i(D_j)$ such that each of the computation takes at most M^2 utility comparisons. How large is this number when $M = 2065$?

(c) Say that $D = (D_1, D_2)$ is a *pseudo Nash equilibrium* iff $D_i \in F_i(D_j)$ for each i and j. Show that every Nash equilibrium is a pseudo Nash equilibrium. Show that there exists pseudo Nash equilibria D^1 and D^2 such that for each pseudo Nash equilibrium D,

$$D_1^1 \geq D_1 \geq D_1^2 \text{ and } D_2^2 \geq D_2 \geq D_2^1;$$

in particular, the above bounds apply for each Nash equilibrium.

(d) Find a procedure for computing D^1 (and D^2) such that there are at most $4M^3$ utility comparisons. Briefly discuss this result in comparing with part (a).

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