Global Games

14.126 Game Theory
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Road map

1. Theory
    1. 2 x 2 Games (Carlsson and van Damme)
    2. Continuum of players (Morris and Shin)
    3. General supermodular games (Frankel, Morris, and Pauzner)
2. Applications
    1. Currency attacks
    2. Bank runs

Motivation

- Outcomes may differ in similar environments.
- This is explained by multiple equilibria (w/strategic complementarity)
  - Investment/Development
  - Search
  - Bank Runs
  - Currency attacks
  - Electoral competition…
- But with introduction of incomplete information, such games tend to be dominance-solvable

A simple partnership game

	Invest	Not-Invest
Invest	θ,θ	$\theta-1, 0$
Not-Invest	$0,\theta-1$	$0,0$

θ is common knowledge

$\theta < 0$

	Invest	Not-Invest
Invest	θ,θ	θ −1, 0
Not-Invest	0,θ−1	0,0

θ is common knowledge

$\theta > 1$

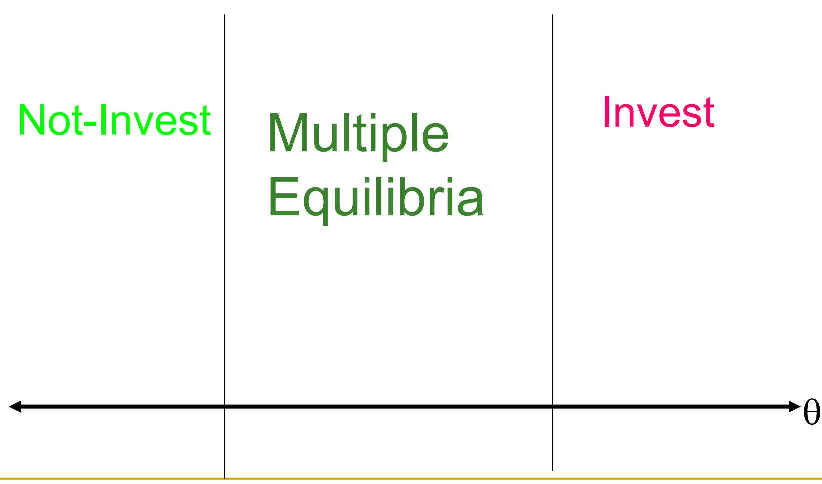
	Invest	NotInvest
Invest	θ,θ	θ −1, 0
NotInvest	0,θ−1	0,0

θ is common knowledge

$0 < \theta < 1$ Multiple Equilibria!!!

	Invest	Not-Invest
Invest	θ,θ	$\theta - 1, 0$
Not-Invest	$0,\theta - 1$	$0,0$

θ is common knowledge



θ is not common knowledge

- θ is uniformly distributed over a large interval
- Each player i gets a signal

$$x_i = \theta + \varepsilon\eta_i$$

  - (η_1, η_2) is bounded,
  - Independent of θ,
  - iid with continuous F (common knowledge),
  - $E[\eta_i] = 0$.

Conditional Beliefs given x_i

$$\theta =_d x_i - \varepsilon\eta_i$$

- i.e. $\Pr(\theta \leq \underline{\theta}|x_i) = 1-F((x_i-\underline{\theta})/\varepsilon)$;

$$x_j =_d x_i + \varepsilon(\eta_j-\eta_i)$$

- $\Pr(x_j \leq x|x_i) = \Pr(\varepsilon(\eta_j - \eta_i) \leq x - x_i)$;
- $F(\theta,x_j|x_i)$ is decreasing in x_i
- $E[\theta|x_i] = x_i$

Payoffs given x_i

	Invest	Not-Inv
Invest	x_i	x_i-1
Not-Inv	0	0

	Invest	Not-Inv
Invest	θ	θ -1
Not-Inv	0	0

- Invest > Not-Invest
- $U_i(a_i,a_j,x_i)$ is supermodular.
- Monotone supermodular!
- There exist greatest and smallest rationalizable strategies, which are
  - Bayesian Nash Equilibria
  - Monotone (isotone)

Monotone BNE

- Best reply:

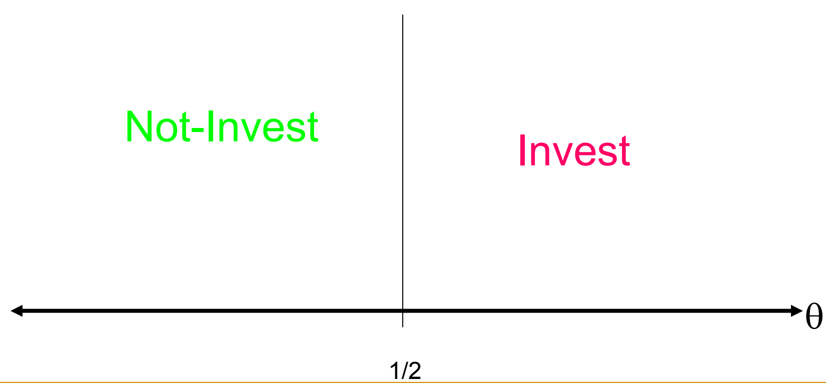
  Invest iff $x_i \geq \Pr(s_j = \text{Not-Invest}|x_i)$
- Assume supp(θ) = [a,b] where $a < 0 < 1 < b$.
- $x_i < 0 \Rightarrow s_i(x_i)$ = Not Invest
- $x_i > 1 \Rightarrow s_i(x_i)$ = Invest
- A cutoff x_i^* s.t.
  - $x_i < x_i^* \Rightarrow s_i(x_i)$ = Not Invest; $x_i > x_i^* \Rightarrow s_i(x_i)$ = Invest;
- Symmetry: $x_1^* = x_2^* = x^*$
- $x^* = \Pr(s_j = \text{Not-Invest}|x^*) = \Pr(x_j < x^*|x_i=x^*) = 1/2$
- "Unique" BNE, i.e., "dominance-solvable"

Questions

- What is the smallest BNE?
- What is the largest BNE?
- Which strategies are rationalizable?
- Compute directly.

θ is not common knowledge but the noise is very small

It is very likely that



Risk-dominance

- In a 2 x 2 symmetric game, a strategy is said to be "risk dominant" iff it is a best reply when the other player plays each strategy with equal probabilities.

	Invest	Not-Invest
Invest	θ,θ	$\theta-1, 0$
Not-Invest	$0,\theta-1$	$0,0$

Invest is RD iff

$$0.5\theta + 0.5(\theta-1) > 0$$

$$\Leftrightarrow \theta > 1/2$$

Players play according to risk dominance!!!

Carlsson & van Damme

Risk Dominance

	A	B
A	u_{11},v_{11}	u_{12},v_{12}
B	u_{21},v_{21}	u_{22},v_{22}

	A	B
A	g_1^a, g_2^a	0,0
B	0,0	g_1^b, g_2^b

- Suppose that (A,A) and (B,B) are NE.
- (A,A) is risk dominant if
  $(u_{11}-u_{21})(v_{11}-v_{12})$
  $> (u_{22}-u_{12})\ (v_{22}-v_{21})$
- $g_1^a = u_{11}-u_{21}$, etc.
- (A,A) risk dominant:
  $g_1^a g_2^a > g_1^b g_2^b$
- i is indifferent against $\underline{s}_j$; (A,A) risk dominant:
  $\underline{s}_1 + \underline{s}_2 < 1$

Dominance, Risk-dominance regions

- Dominance region:

$$D_i^a = \{(u,v) |\ g_i^a > 0,\ g_i^b < 0\}$$

- Risk-dominance region:

$$R^a = \{(u,v) |\ g_1^a > 0\ g_2^a > 0;\ g_1^b,\ g_2^b > 0 \Rightarrow \underline{s}_1 + \underline{s}_2 < 1\}$$

Model

- $\Theta \subseteq \mathfrak{R}^m$ is open; (u,v) are continuously differentiable functions of θ w/ bounded derivatives;
- prior on θ has a density h which is strictly positive, continuously differentiable, bounded.
- Each player i observes a signal

$$x_i = \theta + \varepsilon\eta_i$$

  - (η_1, η_2) is bounded,
  - Independent of θ,
  - Admits a continuous density

Theorem (Risk-dominance v. rationalizability)

- Assume:
  - x is on a continuous curve $C \subseteq \Theta$,
  - $(u(c),v(c)) \in R^a$ for each $c \in C$,
  - $(u(c),v(c)) \in D^a$ for some $c \in C$.
- Then, A is the only rationalizable action at x when ε is small.

"Public" Information

- $\theta \sim N(y,\tau^2)$ and $\varepsilon\eta_i \sim N(0,\sigma^2)$
- Given x_i,

$$\theta \sim N(rx_i+(1-r)y,\ \sigma^2 r)$$
$$x_j \sim N(rx_i+(1-r)y,\ \sigma^2(r+1))$$
$$r = \tau^2/(\sigma^2+\tau^2)$$

- (Monotone supermodularity) monotone symmetric NE w/cutoff x^c:

$$rx^c + (1-r)y = \Pr(x_j \le x^c \mid x_i = x^c) = \Phi\left(\frac{(1-r)(x^c - y)}{\sigma\sqrt{r+1}}\right)$$

- Unique monotone NE (and rationalizable strategy) if

$$rx^c + (1-r)y - \Pr(x_j \le x^c \mid x_i = x^c)$$

is increasing in x^c whenever zero, i.e.,

$$\sigma^2 < 2\pi\tau^4(r+1)$$

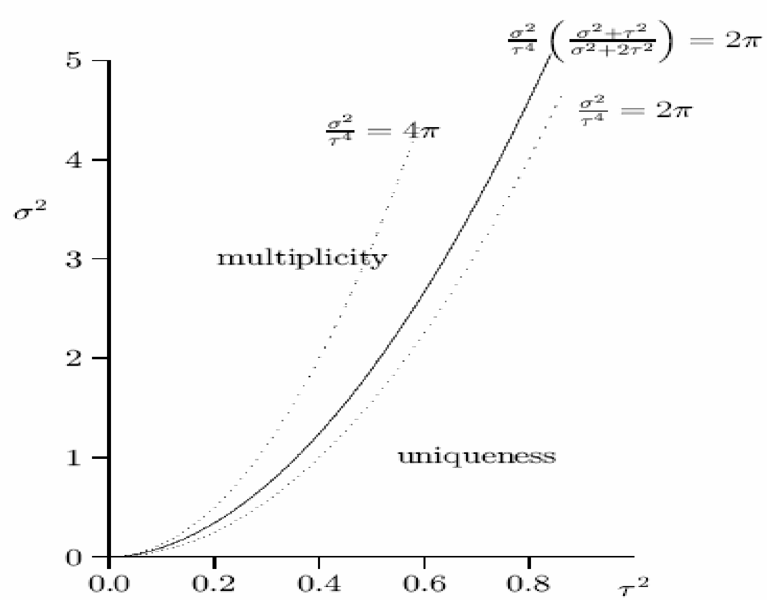


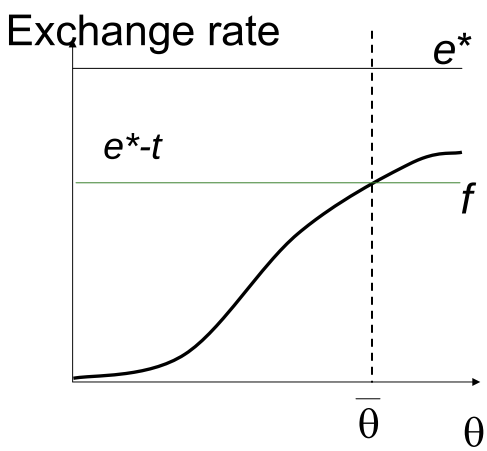
Figure 3.1: Parameter Range for Unique Equilibrium

Currency attacks
Morris & Shin

Model

- Fundamental: θ in [0,1]
- Competitive exchange rate: $f(\theta)$
- f is increasing
- Exchange rate is pegged at $e^* \geq f(1)$.
- A continuum of speculators, who either
  - Attack, which costs t, or
  - Not attack
- Government defends or not
- The exchange rate is e^* if defended, $f(\theta)$ otherwise

Speculator's Payoffs



- Attack, not defended:

  $e^* - f(\theta) - t$
- Attack, defended:

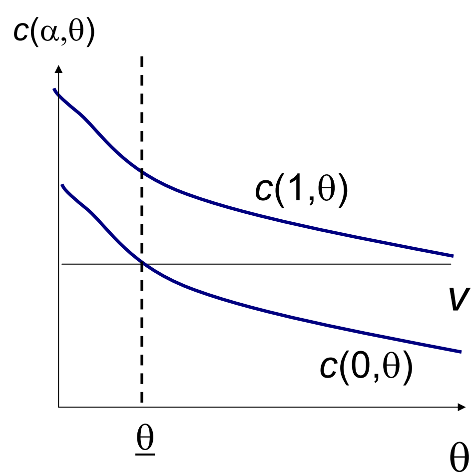
  $-t$
- No attack: 0

Government's payoffs

- Value of peg = v
- Cost of defending

  $c(\alpha,\theta)$

  where α is the ratio of speculators who attack
- c is increasing in α, decreasing in θ



Government's strategy

- Government knows α and θ;
- Defends the peg if

$$v > c(\alpha,\theta)$$

- Abandons it otherwise.

θ is common knowledge

$\theta < \underline{\theta}$

	Attack	NoAttack
Attack	$e^*-f(\theta)-t$	$e^*-f(\theta)-t$
NoAttack	0	0

θ is common knowledge

$\theta > \bar{\theta}$

	Attack	NoAttack
Attack	e*-f(θ)-t	–t
NoAttack	0	0

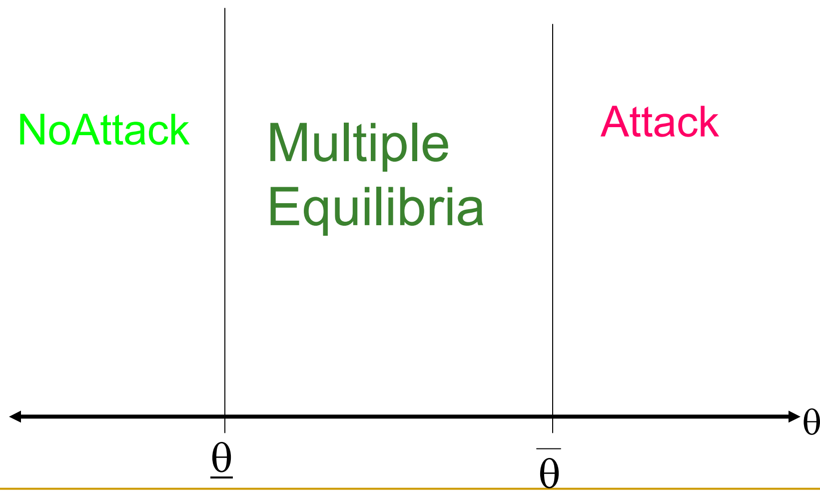
θ is common knowledge

$\underline{\theta} < \theta < \bar{\theta}$

Multiple Equilibria!!!

	Attack	NoAttack
Attack	*e**-*f*(θ)-*t*	–*t*
NoAttack	0	0

θ is common knowledge



θ is not common knowledge

- θ is uniformly distributed on [0,1].
- Each speculator *i* gets a signal

$$x_i = \theta + \eta_i$$

- η_i's are independently and uniformly distributed on $[-\varepsilon, \varepsilon]$ where $\varepsilon > 0$ is very small.
- The distribution is common knowledge.

Government's strategy

- Government knows α and θ;
- Defends the peg if

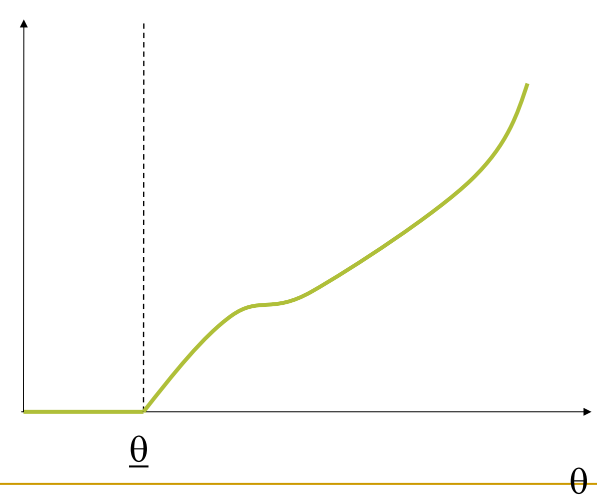
$$v > c(\alpha,\theta)$$

- Abandons it otherwise.

Define: $a(\theta)$ = the minimum α for which G abandons the peg

$$v = c(a(\theta),\theta)$$

$a(\theta)$



Speculator's payoffs

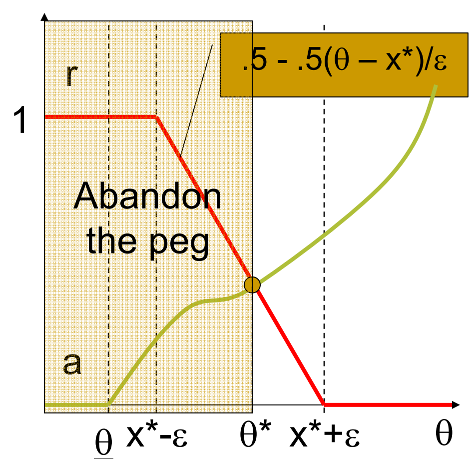
- r = ratio of speculators who attack
- u(Attack,r,θ) = e^* - $f(\theta)$ – t if $r \geq a(\theta)$
  -t otherwise
- U(No Attack,r,θ) = 0

Unique Equilibrium

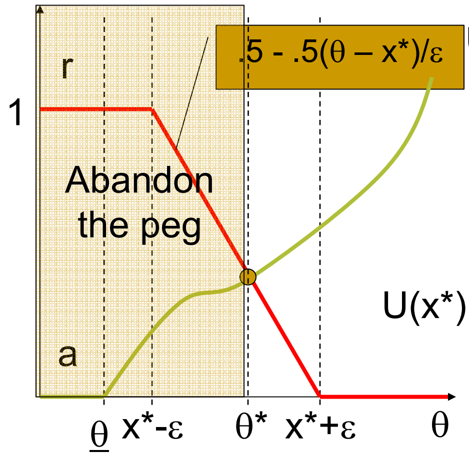
- **Equilibrium:** Attack iff $x_i \leq x^*$.
- $r(\theta) = \Pr(x \leq x^* | \theta)$

$.5 - .5(\theta^* – x^*)/\varepsilon = a(\theta^*)$

$x^* = \theta^* – \varepsilon[1 - 2a(\theta^*)]$



θ*



Utility from attack

$$U(x^*) = \frac{1}{2\varepsilon}\int_{x^*-\varepsilon}^{\theta^*} (e^* - f(\theta))\,d\theta - t$$

$$U(x^*) \approx [\theta^* - x^* + \varepsilon](e^* - f(\theta^*))/(2\varepsilon) - t$$
$$= [1 - a(\theta^*)](e^* - f(\theta^*)) - t$$
$$= 0$$

$$[1 - a(\theta^*)](e^* - f(\theta^*)) = t$$

"Risk dominance"

- Suppose all strategies are equally likely
- *r* is uniformly distributed on [0,1]
- Expected payoff from Attack

$$(1 - a(\theta))(e^* - f(\theta)) - t$$

- Attack is "risk dominant" iff

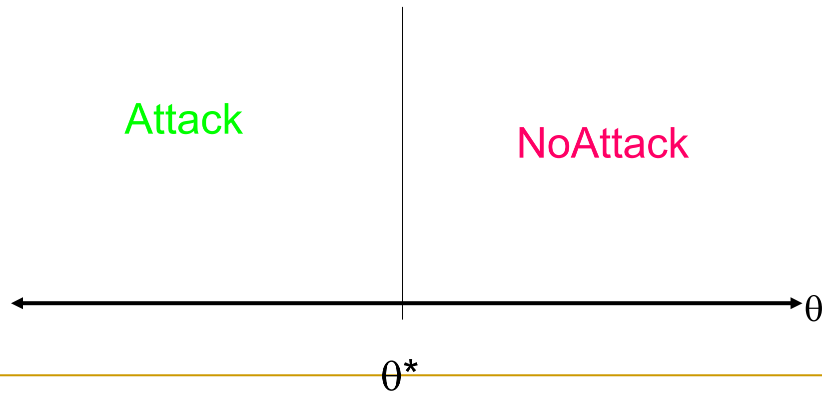
$$(1 - a(\theta))(e^* - f(\theta)) > t$$

- Cutoff value θ*:

$$(1 - a(\theta^*))(e^* - f(\theta^*)) = t$$

θ is not common knowledge but the noise is very small

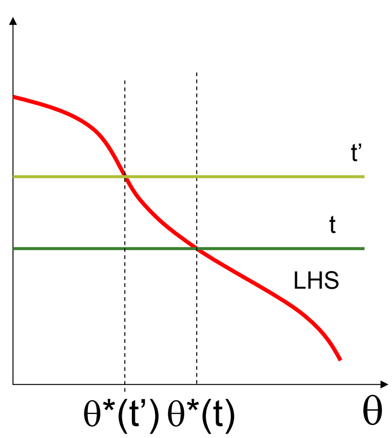
It is very likely that



Comparative statics – t

- Cutoff value θ^*:
  $(1-a(\theta^*))(e^*-f(\theta^*)) = t$
- LHS is decreasing in θ^*.

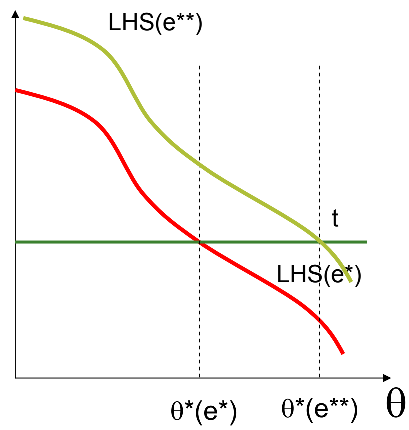
If transaction cost t increases,
attack becomes less likely!



Comparative statics – e*

- Cutoff value θ*:
  (1-a(θ*))(e*-f(θ*)) = t
- LHS is decreasing in θ*
- and increasing in e*

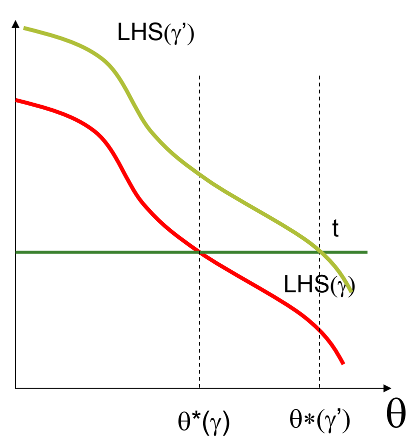
If e* increases, attack becomes more likely!



Comparative statics – c

- Let c(α,θ) = γ C(α,θ)
- Cutoff value θ*:
  (1-a(θ*))(e*-f(θ*)) = t
- LHS is decreasing in θ*
- and decreasing in a
- i.e., increasing in γ

If γ increases, attack becomes more likely!



Bank Runs

Model

- Dates: 0,1,2
- Each depositor has 1 unit good
- A bank invests either in
  - Cash with return 1 at t = 1; or in
  - Illiquid asset (IA) with return R > 1 at t =2.
- Consumption: c_1, c_2
- Two types of depositor
  - Impatient: $\log(c_1)$; measure λ
  - Patient: $\log(c_1+c_2)$; measure $1-\lambda$
- If proportion of L invested in IA withdrawn at t=1, the return is Re^{-L}.

Assume: λ is in cash.

Actions

- An impatient consumer withdraws at t=1.
- A (patient) consumer either withdraws at t=1 and gets 1 unit of cash, with payoff

$$u(1) = \log(1) = 0,$$

- or withdraws at t =2 and gets Re^{-L} where L is the ratio of patient consumers who withdraws at t=1.
- Write $r = \log(R)$.
- The payoff from late withdrawal is

$$u(2) = r - L.$$

Complete Information

- Multiple equilibria:
- All patients consumers withdraw at t = 2, where L = 0.
- All patients consumers withdraw at t = 1, where L = 1.

Incomplete Information

- r is distributed with $N(\underline{r}, 1/\alpha)$, where

$$0 < \underline{r} < 1.$$

- Each depositor i gets a signal

$$x_i = r + \varepsilon_i$$

- ε_i iid with $N(0, 1/\beta)$.
- The distribution is common knowledge.

This is identical to the partnership game!!
(when $\beta \to \infty$)

Theorem

- Write $\rho = (\alpha\underline{r}+\beta x)/(\alpha+\beta)$ for the expected value of r given x.
- Write $\gamma = \alpha^2(\alpha+\beta)/(\alpha\beta+2\beta^2)$.
- If $\gamma < 2\pi$, there is a unique equilibrium; a patient depositor withdraws at t = 1 iff $\rho < \rho^*$, where

$$\rho^* = \Phi(\gamma^{.5}(\rho^* - \underline{r})).$$

General Supermodular Global Games
Frankel, Morris, and Pauzner

Model

- $N = \{1,\ldots,n\}$ players
- $A_i \subseteq [0,1]$,
  - countable union of closed intervals
  - $0,1 \in A_i$
- Uncertain payoffs $u_i(a_i,a_{-i},\theta)$
  - continuous with bounded derivatives
- 1-dimensional payoff uncertainty: $\theta \in R$
- Each player *i* observes a signal

$$x_i = \theta + \varepsilon\eta_i$$

  - (θ, η_1, η_2) are independent with atomless densities
  - (η_1, η_2) bounded

Main Assumptions

Let $Du_i(a_i,a'_i,a_{-i},\theta) = u_i(a_i,a_{-i},\theta) - u_i(a'_i,a_{-i},\theta)$

- Strategic complementarities: $a_i \geq a'_i$ & $a_{-i} \geq a'_{-i}$ $\Rightarrow Du_i(a_i,a'_i,a_{-i},\theta) \geq Du_i(a_i,a'_i,a'_{-i},\theta)$
- Dominance regions:
  - 0 is dominant when θ is very small
  - 1 is dominant when θ is very large
- State monotonicity: outside dominance regions, $\exists K>0$: $\forall\ a_i \geq a'_i\ \forall\ \theta \geq \theta'$,

$$Du_i(a_i,a'_i,a_{-i},\theta) - Du_i(a_i,a'_i,a_{-i},\theta') \geq K(a_i - a'_i)(\theta - \theta')$$

Theorem (Limit Uniqueness)

- In the limit $\varepsilon \to 0$, there is a "unique" rationalizable strategy, which is increasing.
- i.e., there exists an increasing pure strategy profile s^* such that if for each $\varepsilon > 0$, s^ε is rationalizable at ε, then almost everywhere

$$\text{Lim}_{\varepsilon\to 0}\ s_i^\varepsilon(x_i) = s_i^*(x_i).$$

Limit Solution

- $(s_1^*(x), s_2^*(x))$ is a Nash equilibrium of the complete information game in which it is common knowledge that $\theta = x$.

Noise dependence

- There exists a game satisfying the FPM assumptions in which for different noise distributions, different equilibria are selected in the limit as the signal errors vanish.
- There are conditions under which s* is independent of the noise distributions.