In class we used to following property twice (for the functions V and F when trying to construct a Liapunov function for a concave discounted dynamic program). Some people later asked me where this comes from.

Lemm(it)a. Suppose we have some function $H(z)$ that is concave and differentiable on $Z \subset R^N$ then we want to prove:

$$(H'(z) - H'(z')) \cdot (z - z') \leq 0.$$

for all $z, z' \in Z$.

Proof. By concavity of H we have that

$$H'(z') \cdot (z - z') \geq H(z) - H(z')$$

and inverting the roles of z and z'

$$H'(z) \cdot (z' - z) \geq H(z') - H(z)$$

The result follows by adding up both inequalities and rearranging.

Remark. Clearly if H is strictly concave the inequality is strict for $z' \neq z$.

Remark. For $N = 1$ there is an intuitive graph that exemplifies this inequality. Note that we are not assuming $N = 1$ however (above $H'(z)$ and z are vectors)