Example 1. Consider the following simple function

$$h(x) = -x^3 + x$$

then clearly $\bar{x} = 0$ is the unique steady state of $x_{t+1} = h(x_t)$. It is also globally stable. This follows since $-x^3 < 0$ for $x > 0$ and $-x^3 > 0$ for $x < 0$ so that $h(x) = -x^3 + x < x$ for $x > 0$ and $h(x) = -x^3 + x > x$ for $x < 0$. Thus x is rising if x is below 0 and falling if it is above 0. The convergence is then monotonic.

However note that at $\bar{x} = 0$ we have that $h'(0) = 1$ so that $A = 1$ and the eigenvalue is $\lambda = 1$, thus $|\lambda| = 1$.

One may oppose this example since we were requiring $I - A$ to be non-singular, here $I - A = 0$ so it is singular. The next example shows a case with $I - A$ non-singular.

Example 2. Take

$$h(x) = x^3 - x$$

it is easy to see that $\bar{x} = 0$ is the unique steady state of $x_{t+1} = h(x_t)$ for $x \in [-1, 1]$. It is easy to see that the system is locally stable around $\bar{x}$ (it is not montonic though).

However note that at $\bar{x} = 0$ we have that $h'(0) = -1$ so that $A = -1$ and the eigenvalue is $\lambda = -1$, thus $|\lambda| = 1$. Note that in this case $I - A = -2$ is singular.

Note: Clearly an eigenvalue with absolute value of 1 does not ensure local convergence, just take $h(x) = x^3 + x$ or $h(x) = -x^3 - x$ for example.

Remarks: Of course both of these policy functions can be generated as *optimal* policy functions for some concave F and some $0 < \beta < 1$ using the Boldrin-Montrucchio construction argument we went over in class. Thus these point are of interest for us, they can arise in applications.

We conclude from these 2 examples that a one dimensional system *may* be stable even if we don't have $|\lambda| < 1$, if we do have $|\lambda| = 1$. More generally, with more dimensions this point *may* affect the dimensionality of the subset of the neighbourhood over which the system is stable. That is, even if we have $|\lambda_i| < 1$ for only m eigenvalues, if we have some other eigenvalues with $|\lambda_i| = 1$ we *may* [we can't be sure, see the "note" above] have convergence starting from x_0 belonging to a subset of greater dimension than m.