Problem Set #2: Recursive Methods

Spring 2003

1 Differentiability of the value function

This problem is for those that would like to attempt it. There is no need to hand it in.

For any dynamic program show that the value function, $v(\cdot)$, is differentiable at any point y that is an interior optimum for some $x \in X$, i.e. $y \in int(\Gamma(x))$ [and thus $y \in int(X)$] and $y \in G(x)$ where G is the optimal policy correspondence, show that in this case $v'(y) = F_x(y, g(y))$ (the usual formula). Is this result useful? Why or why not?

(Hint: use the fact that the Euler equations are necessary for an interior optimum)

2 Numerical exercise: Neoclassical Growth Model Extensions

You can write your own code or modify the code I sent you. Throughout, use a grid for k that is at least between .01 and 1.2 of the usual positive steady state for capital, k_{ss} (for the second problem you can set the lower bound of your capital grid to 0 since output is still positive there). Use at least 1000 points in your grid.

2.1 Elastic Labor Supply

Solve the following problem by value function iteration using a grid over capital:

$$\max_{\{c_t,n_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

$$\begin{aligned} c_t + k_{t+1} &\leq G(k_t, n_t) + (1-\delta) k_t \\ c_t &\geq 0,\ 1 \geq n_t \geq 0,\ k_{t+1} \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

with

$$\begin{aligned} u(c, 1-n) &= \log(c) + \gamma \log(1-n) \\ G(k, n) &= k^{\alpha} n^{1-\alpha} \end{aligned}$$

[Hints: 1. compute first the steady state and then set the appropriate grid for k [with the preferences and technology you should be able to compute this exactly in the following recursive way: get k/n from the steady state Euler equation, then get c/n from the resource constraint, finally get c from the f.o.c. between c and n]. *Then "max-out" labor for given k_t and k_{t+1}, computing the function (in your numerical approach this will be a matrix of course):*

$$W(k, k') \equiv \max_{c,n} u(c, 1-n)$$

$$\begin{aligned} c &\leq G(k, 1-n) + (1-\delta) k - k' \\ c &\geq 0,\ 1 \geq n \geq 0 \end{aligned}$$

[indeed, can you find a closed form solution to this problem by hand or must you compute W numerically somehow?]. Armed with W *this allows you to solve the iteration part exactly as in the model without labor. What is the right bounds for* k' given k? That is, what is Γ?*]*

You may use any parameterization you like but try to find one for which the results look good.

(a) Plot the resulting policy functions for k' (against the 45 degree line), and for c and n. Discuss.

(b) Compute a sample path starting from $k_0 = .5 k_{ss}$ where k_{ss} is the steady state capital level.

2.2 Neoclassical Growth Model with a Non-concave Production Function: Poverty Traps

Consider the standard optimal growth model:

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} c_t + k_{t+1} &\leq f(k_t) + (1-\delta) k_t \\ c_t &\geq 0,\ k_{t+1} \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

with

$$\begin{aligned} u(c) &= \frac{c^{1-\sigma}}{1-\sigma} \\ f(k) &= \max\{k^{\alpha}, w + Rk\} \end{aligned}$$

for $w > 0$ and $\beta^{-1} > R > 1$. The only twist here is that f is not globally concave, it has a convex part to it (at the kink).

You must solve this problem numerically to discuss your results. Use throughout the following parameters:

$$\begin{aligned} \alpha &= .33 \\ \sigma &= 2 \\ \delta &= 1 \\ \rho &= .03 \text{ used to define } \beta \text{ and } R \\ R &= 1/(1+\rho) \\ R &= 1 + \frac{\rho}{5} \end{aligned}$$

And three different values for w which we specified below.

Case 1 $w = 0.38259402192557$ [**medium w**]

(a) What does the optimal G correspondence look like? Plot the optimal correspondence for k' against k together with a 45 degree line. Note that numerically you may be focusing on a policy *function*. But argue that for this calibration there is one "jump" in the policy "function" at a point $\hat{k}$, and that for this $\hat{k}$ there are actually two optimal choices for k' [hint: use the Theorem of the Maximum]. Conclude then that the optimal policy correspondence is not a function. Does $\hat{k}$ necessarily coincide with the inflexion (and kink) point of the production function (i.e. with $\tilde{k}$ such that $\tilde{k}^{\alpha} = w + R\tilde{k}$)?

(b) Show that there are "poverty traps" by showing that the long run level of k depends on the initial capital level k_0.

(c) Plot the optimal policy for consumption. At $\hat{k}$ what are the relative merits in terms of the resulting sequence for consumption of the two optimal paths?

(c) Plot the value function $v(k)$ to help you answer the following. Is $v(k)$ concave at $\hat{k}$? What can be said about the differentiability of the value function at $\hat{k}$? Is $v(k)$ locally concave for all $k \neq \hat{k}$? [hint: take my word that $v(k)$ is differentiable for $k \neq \hat{k}$ in this case, use the envelope condition and your policy correspondence for c].

Case 2 $w = 0.38116063066242$ [**low w**]

(a) Plot the optimal correspondence for k' against k together with a 45 degree line. What does the optimal G correspondence look like? How many "jumps" do you know find in your optimal policy? Can you find any intuition for these "jumps"? [hint: think about a capital level that would lead you to the kink $\tilde{k}$ in the production function] What can be said then about differentiability at these points?

(b) What do the dynamics look like from any k_0? Are there "poverty traps" now?

Case 3 $w = 0.38313637477680$ [**high w**]

(a) Plot the optimal correspondence for k' against k together with a 45 degree line. What do the dynamics look like from any k_0? Are there "poverty traps" now?