1 Solutions Pset 3

1) Do some programing

3) Brock Mirman problem

a) Take $V = a_1 \log k + a_2 \log \theta + a_3$. Then the max problem is

$$\begin{aligned}
TV(k) &= \max_{0 \leq k' \leq Ak^\alpha \theta} \ln(Ak^\alpha \theta - k') + \beta E_\theta [a_1 \log k' + a_2 \log \theta + a_3] \\
TV(k) &= \max_{0 \leq k' \leq Ak^\alpha \theta} \ln(Ak^\alpha \theta - k') + \beta a_1 \log k' + \beta a_2 E_\theta \log \theta + \beta a_3
\end{aligned}$$

The FOC condition for this problem is (assuming interior),

$$-\frac{1}{Ak^\alpha \theta - k'} + \frac{\beta a_1}{k'} = 0$$

Which implies that

$$k' = \frac{\beta a_1}{1 + \beta a_1} Ak^\alpha \theta$$

And

$$\begin{aligned}
TV(k) &= \ln\left(\frac{1}{1 + \beta a_1} Ak^\alpha \theta\right) + \beta a_1 \log \frac{\beta a_1}{1 + \beta a_1} Ak^\alpha \theta \\
&\quad + \beta a_2 E_\theta \log \theta + \beta a_3 \\
&= \alpha(1 + \beta a_1) \log k + (1 + \beta a_1) \log \theta + \\
&\quad + \left[\beta a_1 \log \frac{A \beta a_1}{1 + \beta a_1} + \ln\left(\frac{A}{1 + \beta a_1}\right) + \beta a_2 E_\theta \log \theta + \beta a_3\right]
\end{aligned}$$

Given that $V(k) = a_1 \log k + a_2 \log \theta + a_3$, using $V(k) = TV(k)$ we have that

$$\begin{aligned}
a_1 &= \alpha(1 + \beta a_1) \\
a_2 &= 1 + \beta a_1 \\
a_3 &= \beta a_1 \log \frac{A \beta a_1}{1 + \beta a_1} + \ln\left(\frac{A}{1 + \beta a_1}\right) + \beta a_2 E_\theta \log \theta + \beta a_3
\end{aligned}$$

So,

$$\begin{aligned}
a_1 &= \frac{\alpha}{1 - \beta \alpha} > 0 \\
a_2 &= \frac{1}{1 - \beta \alpha}
\end{aligned}$$

And a_3 is given by

$$a_3 = \frac{1}{1-\beta}\left[\beta a_1 \log \frac{A\beta a_1}{1+\beta a_1} + \ln\left(\frac{A}{1+\beta a_1}\right) + \beta a_2 E_\theta \log \theta\right]$$

The proof that $V = V^*$ is done in page 275,276 of SLP.

b) The optimal rule for consumption is then

$$\begin{aligned} c(k,\theta) &= Ak^\alpha\theta - k'(k,\theta) = \\ c(k,\theta) &= \frac{1}{1+\beta a_1} Ak^\alpha \theta \\ c(k,\theta) &= (1-\beta\alpha) Ak^\alpha\theta \end{aligned}$$

So, we have that

$$\frac{\partial c}{\partial \beta} < 0$$

and

$$\begin{aligned} \frac{\partial c}{\partial \alpha} &= \alpha(1-\beta\alpha) Ak^{\alpha-1}\theta - \beta Ak^\alpha \theta \\ &= [\alpha(1-\beta\alpha) - \beta k] Ak^{\alpha-1}\theta \end{aligned}$$

There are two effects, depending on the level of k.

c)

You can do it ex-ante (before the value of θ is realized), then

$$V(k) = \int \left(\max_{0<k'\leq Ak^\alpha\theta} \{\ln(Ak^\alpha\theta - k') + \beta V[k']\}\right) h(\theta)\, d\theta$$

4)

a) The main conflict is the change in preferences. They value consumption paths differentely because they discount the future in different ways. In particular, time-t self values consumption at time-t versus time-$(t+1)$more than any time-τ self with $\tau < t$, as long as $\beta < 1$. For $\beta = 1$ they all agree.

b) Every self maximizes its utility subject to what other types will do in the future. So,

$$V(k_0) = \max_c u(c) + \delta W(k_1) \tag{1}$$

Where $\delta W(k)$ is the discounted value for todays self of leaving k' for the future. So,

$$W(k_t) = \beta \sum_i \delta^i u(c^*(k_{t+i}))$$

Where $c^*(k_{t+i})$ is the optimal consumption rule that future selfs will follow (we are assuming symmetry, and hence c^* is time-independent). Now take (**??**) and do the following :

$$\begin{aligned} V(k_0) &= u(c_0^*) + \delta W(k_1) = u(c_0^*) + \beta\delta \sum_i \delta^i u(c^*(k_{t+i})) \\ V(k_0) - (1-\beta) u(c_0^*) &= \beta u(c_0^*) + \beta\delta \sum_i \delta^i u(c^*(k_{t+i})) \\ V(k_0) - (1-\beta) u(c_0^*) &= W(k_0) \end{aligned}$$

So, We can define W recursevly as

$$\begin{aligned} W(k) &= V(k) - (1-\beta) u(c^*(k)) \\ W(k) &= \max_c \{u(c) + \delta W(f(k) - c)\} - (1-\beta) u(c^*(k)) \end{aligned}$$

The T operator is such that $TW(k) = \max_c \{u(c) + \delta W(f(k) - c)\} - (1-\beta) u(c^*(k))$ and we are looking for a fixed point of T.

c) If $\beta = 1$, you can easily show that T is a contraction mapping (is monotone and satisfies discounting). This means that there is a unique W that solves the functional equation, and unique Markov equilibrium.

d) If $\beta < 1$ the T operator satisfies discounting :

$$\begin{aligned} T(W(k) + a) &= \max_c \{u(c) + \delta (W(f(k) - c) + a)\} - (1-\beta) u(c^*(k)) \\ &= \max_c \{u(c) + \delta W(f(k) - c)\} - (1-\beta) u(c^*(k)) + \delta a \\ &= TW(k) + \delta a \end{aligned}$$

It does not however, necessarly satisfies monotonicity. Higher W, might imply higher $c^*(k)$ for some capital level, and hence $\max_c \{u(c) + \delta (W(f(k) - c) + a)\} - (1-\beta) u(c^*(k))$ might not increase.

e) If $u = \log c$ and $f = Ak^\alpha$, then we can do part 3.

$$\begin{aligned} TW(k) &= \max_c \{u(c) + \delta W(f(k) - c)\} - (1-\beta) u(c^*(k)) \\ &= \max_c \{\log c + \delta a \log(Ak^\alpha - c) + \delta b\} - (1-\beta) u(c^*(k)) \end{aligned}$$

$$\begin{aligned} c^*(k) &: \\ \frac{1}{c} &= \frac{\delta a}{Ak^\alpha - c} \\ c &= \frac{1}{1+\delta a} Ak^\alpha \end{aligned}$$

So,

$$\begin{aligned}
TW(k) &= \log \frac{1}{1+\delta a} A k^{\alpha} + \delta a \log \left(A k^{\alpha} - \frac{1}{1+\delta a} A k^{\alpha} \right) \\
&\quad + \delta b - (1-\beta) \log \frac{1}{1+\delta a} A k^{\alpha} \\
&= \log \frac{1}{1+\delta a} A + \alpha \log k + \delta A \log \frac{\delta a}{1+\delta a} A + \alpha \delta a \log k + \\
&\quad + \delta b - (1-\beta) \log \frac{1}{1+\delta a} A - (1-\beta) \alpha \log k \\
&= \alpha \left[(1+\delta a) - (1-\beta) \right] \log k + \delta b + \log \frac{1}{1+\delta a} A \\
&\quad + \delta A \log \frac{\delta a}{1+\delta a} A - (1-\beta) \log \frac{1}{1+\delta a} A
\end{aligned}$$

So,

$$a = \frac{\alpha\beta}{1-\alpha\delta}$$

And you can easily compute b.
The equilibrium consumption policy is then

$$c = \frac{1-\alpha\delta}{1-\alpha\delta(1-\beta)} A k^{\alpha}$$

Higher β implies higher consumption (the impatience has decreased).
f) For $\beta = 0$ we have that

$$\tilde{c} = (1 - \alpha\delta^{e}) A k^{\alpha}$$

So we need $\tilde{\delta}$ to be such that

$$\frac{1}{1-\alpha\delta(1-\beta)} \beta\delta = \delta^{e}$$

Now

$$\delta > \delta^{e} > \beta\delta$$

given that $\beta < 1$.

A hyperbolic consumer looks like an exponential with an appropiate discount rate!!.

Exercise 6.7

a. Actually, Assumption 4.9 is not needed for uniqueness of the optimal capital sequence.

A4.3: $K = [0,1] \subseteq R^l$ and the correspondence

$$\Gamma(k) = \{y : y \in K\}$$

is clearly compact-valued and continuous.

A4.4: $F(k,y) = (1-y)^{(1-\theta)\alpha} k^{\theta\alpha}$ is clearly bounded in K, and it is also continuous. Also, $0 \leq \beta \leq 1$.

A4.7: Clearly F is continuously differentiable, then

$$\begin{aligned}
F_k &= \theta\alpha(1-y)^{(1-\theta)\alpha} k^{\theta\alpha-1} \\
F_y &= -(1-\theta)\alpha(1-y)^{(1-\theta)\alpha-1} k^{\theta\alpha} \\
F_{kk} &= \theta\alpha(1-y)\left(\theta\alpha-1\right)^{(1-\theta)\alpha} k^{\theta\alpha-2} < 0 \\
F_{yy} &= (1-\theta)\alpha[(1-\theta)\,\alpha-1](1-y)^{(1-\theta)\alpha-2} k^{\theta\alpha} < 0 \\
F_{xy} &= -\theta\alpha\,(1-\theta)\,\alpha(1-y)^{(1-\theta)\alpha-1} k^{\theta\alpha-1} < 0,
\end{aligned}$$

and $F_{kk}F_{yy} - F_{xy}^2 > 0$, hence F is strictly concave.

A4.8: Take two arbitrary pairs (k,y) and (k',y') and $0 < \pi < 1$. Define $k^\pi = \pi k + (1-\pi)k'$, $y^\pi = \pi y + (1-\pi)y'$. Then, since $\Gamma(k) = \{y : 0 \leq y \leq 1\}$ for all k it follows trivially that if $y \in \Gamma(k)$ and $y' \in \Gamma(k')$ then $y^\pi \in \Gamma(k^\pi) = \Gamma(k) = \Gamma(k') = K$.

A4.9: Define $A = K \times K$ as the graph of Γ. Hence F is continuously differentiable because U and f are continuously differentiable. The Euler equation is

$$\alpha(1-\theta)(1-k_{t+1})^{(1-\theta)\alpha-1} k_t^{\theta\alpha} = \beta\alpha\theta\,(1-k_{t+2})^{(1-\theta)\alpha}\, k_{t+1}^{\theta\alpha-1}.$$

b. Evaluating the Euler equation at $k_{t+1} = k_t = k^*$, we get

$$(1-\theta)k^* = \beta\theta\left(1-k^*\right),$$

or

$$k^* = \frac{\beta\theta}{1-\theta+\beta\theta}.$$

c. From the Euler equation, define

$$\begin{aligned} & W(k_t, k_{t+1}, k_{t+2}) \\ \equiv\; & \alpha(1-\theta)(1-k_{t+1})^{(1-\theta)\alpha-1}k_t^{\theta\alpha} \\ & -\beta\alpha\theta\left(1-k_{t+2}\right)^{(1-\theta)\alpha}k_{t+1}^{\theta\alpha-1} \\ =\; & 0. \end{aligned}$$

Hence, expanding W around the steady state

$$\begin{aligned} W(k_t, k_{t+1}, k_{t+2}) &= W(k^*, k^*, k^*) + W_1(k^*)\left(k_t - k^*\right) \\ &\quad + W_2(k^*)\left(k_{t+1} - k^*\right) + W_3(k^*)\left(k_{t+2} - k^*\right), \end{aligned}$$

where

$$\begin{aligned} W_1(k^*) &= \alpha^2(1-\theta)\theta(1-k^*)^{(1-\theta)\alpha-1}\left(k^*\right)^{\theta\alpha-1}, \\ W_2(k^*) &= -\alpha(1-\theta)\left[(1-\theta)\alpha-1\right]\left(1-k^*\right)^{(1-\theta)\alpha-2}\left(k^*\right)^{\theta\alpha} \\ &\quad -\beta\theta\alpha(\theta\alpha-1)\left(1-k^*\right)^{(1-\theta)\alpha}\left(k^*\right)^{\theta\alpha-2}, \\ W_3(k^*) &= \beta\theta\alpha^2(1-\theta)\left(1-k^*\right)^{(1-\theta)\alpha-1}\left(k^*\right)^{\theta\alpha-1}. \end{aligned}$$

Normalizing by $W_3(k^*)$ and using the expression obtained for the steady state capital we finally get

$$\beta^{-1}\left(k_t - k^*\right) + B\left(k_{t+1} - k^*\right) + \left(k_{t+2} - k^*\right) = 0,$$

where

$$B = \frac{1-\alpha(1-\theta)}{\alpha(1-\theta)} + \frac{1-\alpha\theta}{\alpha\theta\beta}.$$

That both of the characteristic roots are real comes from the fact that the return function satisfies Assumptions 4.3-4.4 and 4.7-4.9 and it is twice differentiable, so the results obtained in Exercise 6.6 apply.

To see that $\lambda_1 = (\beta\lambda_2)^{-1}$ it is straightforward from the fact that

$$\begin{aligned}
\lambda_1\lambda_2 &= \left(\frac{(-B)+\sqrt{B^2-4\beta^{-1}}}{2}\right)\left(\frac{(-B)-\sqrt{B^2-4\beta^{-1}}}{2}\right) \\
&= \frac{(-B)^2-(B^2-4\beta^{-1})}{4} \\
&= \beta^{-1}.
\end{aligned}$$

To see that $\lambda_1 + \lambda_2 = -B$, just notice that

$$\lambda_1+\lambda_2 = \frac{(-B)+\sqrt{B^2-4\beta^{-1}}}{2} + \frac{(-B)-\sqrt{B^2-4\beta^{-1}}}{2} = -B.$$

Then, $\lambda_1\lambda_2 > 0$ and $\lambda_1+\lambda_2 < 0$ implies that both roots are negative.

In order to have a locally stable steady state k^* we need one of the characteristic roots to be less than one in absolute value. Given that both roots are negative, this implies that we need $\lambda_1 > -1$, or

$$-B+\sqrt{B^2-4\beta^{-1}} > -2,$$

which after some straightforward manipulation implies

$$B > \frac{1+\beta}{\beta}.$$

Substituting for B we get

$$\frac{1-\theta+\theta\beta}{2\theta(1+\beta)(1-\theta)} > \alpha,$$

or equivalently

$$\beta > \frac{(2\theta\alpha-1)(1-\theta)}{[1-2\alpha(1-\theta)]\theta}.$$

d. To find that $k^* = 0.23$, evaluate the equation for k^* obtained in b. at the given parameter values. To see that k^* is unstable, evaluate λ_1 at the given parameter values. Notice also that those parameter values do not satisfy the conditions derived in c.

e. Note that since F is bounded, the two-cycle sequence satisfies the transversality conditions

$$\begin{aligned} \lim_{t\to\infty} \beta^t F_1(x,y)\cdot x &= 0 \quad \text{and} \\ \lim_{t\to\infty} \beta^t F_1(x,y)\cdot y &= 0, \end{aligned}$$

for any two numbers $x, y \in [0,1]$, $x \neq y$. Hence, by Theorem 4.15, if the two cycle (x, y) satisfies

$$\begin{aligned} F_y(x,y) + \beta F_x(y,x) &= 0 \quad \text{and} \\ F_y(y,x) + \beta F_x(x,y) &= 0, \end{aligned}$$

it is an optimal path.

Conversely, if (x, y) is optimal and the solution is interior, then it satisfies

$$\begin{aligned} F_y(x,y) + \beta v'(y) = 0 \quad &\text{and} \quad v'(y) = F_x(y,x), \\ F_y(y,x) + \beta v'(x) = 0 \quad &\text{and} \quad v'(x) = F_x(x,y), \end{aligned}$$

and hence it satisfies the Euler equations stated in the text.

Notice that the pair (x, y) defining the two-cycle should be restricted to the open interval $(0, 1)$.

f. We have that

$$\begin{aligned} F_y(x,y) + \beta F_x(y,x) &= \beta\alpha\theta y^{\alpha\theta-1}(1-x)^{\alpha(1-\theta)} \\ &\quad -\alpha(1-\theta)x^{\alpha\theta}(1-y)^{\alpha(1-\theta)-1}, \end{aligned}$$

and

$$\begin{aligned} F_y(y,x) + \beta F_x(x,y) &= \beta\alpha\theta x^{\alpha\theta-1}(1-y)^{\alpha(1-\theta)} \\ &\quad -\alpha(1-\theta)y^{\alpha\theta}(1-x)^{\alpha(1-\theta)-1} \end{aligned}$$

wrong numbers

The pair ~~(.29, 0.18)~~ the above set of equations equal to zero, and from the result proved in part e. we already know this is a necessary and sufficient condition for the pair to be a two-cycle.

g. Define

$$
\begin{aligned}
E^1(k_t, k_{t+1}, k_{t+2}, k_{t+3}) &\equiv -\alpha(1-\theta){k_{t+1}}^{\alpha\theta}(1-k_{t+2})^{\alpha(1-\theta)-1} \\
&\quad +\beta\alpha\theta k_{t+2}^{\alpha\theta-1}(1-k_{t+3})^{\alpha(1-\theta)} \\
&= -\alpha(1-\theta)x^{\alpha\theta}(1-y)^{\alpha(1-\theta)-1} \\
&\quad +\beta\alpha\theta y^{\alpha\theta-1}(1-x)^{\alpha(1-\theta)} \\
&= 0 \\
E^2(k_t, k_{t+1}, k_{t+2}, k_{t+3}) &\equiv -\alpha(1-\theta){k_t}^{\alpha\theta}(1-k_{t+1})^{\alpha(1-\theta)-1} \\
&\quad +\beta\alpha\theta k_{t+1}^{\alpha\theta-1}(1-k_{t+2})^{\alpha(1-\theta)} \\
&= -\alpha(1-\theta)y^{\alpha\theta}(1-x)^{\alpha(1-\theta)-1} \\
&\quad +\beta\alpha\theta x^{\alpha\theta-1}(1-y)^{\alpha(1-\theta)} \\
&= 0.
\end{aligned}
$$

Let E_i^j be the derivative of E^j with respect to the i^{th} argument. Then, the derivatives are

$$
\begin{aligned}
E_1^1 &= 0, \\
E_2^1 &= -\alpha^2\theta(1-\theta)x^{\alpha\theta-1}(1-y)^{\alpha(1-\theta)-1}, \\
E_3^1 &= -\alpha(1-\theta)x^{\alpha\theta}[\alpha(1-\theta)-1](1-y)^{\alpha(1-\theta)-2} \\
&\quad +\beta\alpha\theta(\alpha\theta-1)y^{\alpha\theta-2}(1-x)^{\alpha(1-\theta)}, \\
E_4^1 &= \beta\alpha\theta y^{\alpha\theta-1}(1-x)^{\alpha(1-\theta)-1}, \\
E_1^2 &= -\alpha^2\theta(1-\theta)y^{\alpha\theta-1}(1-x)^{\alpha(1-\theta)-1}, \\
E_2^2 &= -\alpha(1-\theta)y^{\alpha\theta}[\alpha(1-\theta)-1](1-x)^{\alpha(1-\theta)-2} \\
&\quad +\beta\alpha\theta(\alpha\theta-1)x^{\alpha\theta-2}(1-y)^{\alpha(1-\theta)}, \\
E_3^2 &= \beta\alpha\theta x^{\alpha\theta-1}(1-y)^{\alpha(1-\theta)-1}, \\
E_4^2 &= 0.
\end{aligned}
$$

Using the fact that $k_{t+2} = k_t$ in E_1, expand this system around (0.29,0.18). Denoting by $\hat{K}$ deviations around the stationary point $\bar{K}$, we can express the linearized system as

$$
\hat{K}_{t/2+1} = \begin{bmatrix} \hat{k}_{t+3} \\ \hat{k}_{t+2} \end{bmatrix} = \hat{H} \begin{bmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{bmatrix} = \hat{H}\hat{K}_{t/2}
$$

where

$$
\hat{H} = \begin{bmatrix} E_4^1 & 0 \\ 0 & E_3^2 \end{bmatrix}^{-1} \begin{bmatrix} E_2^1 & E_1^1 \\ E_2^2 & E_1^2 \end{bmatrix}
$$

evaluate this to get stability