Recursive Methods

Outline Today's Lecture

- continue APS:

  worst and best value
- Application: Insurance with Limitted Commitment
- stochastic dynamics

B(W) operator

Definition: For each set $W \subset R$, let $B(W)$ be the set of possible values $\omega = (1-\delta)r(x,y)+\delta\omega_1$ associated with some admissible tuples $(x, y, \omega_1, \omega_2)$ wrt W :

$$B(W) \equiv \left\{ w : \begin{array}{l} \exists\,(x,y) \in C \text{ and } \omega_1, \omega_2 \in W \text{ s.t.} \\ (1-\delta)r(x,y) + \delta\omega_1 \geq (1-\delta)r(x,\hat{y}) + \delta\omega_2,\ \forall \hat{y} \in Y \end{array} \right\}$$

- note that V is a fixed point $B\,(V) = V$
- actually, V is the biggest fixed point

  [fixed point not necessarily unique!]

Finding V

In this simple case here we can do more...

- lowest v is self-enforcing

  highest v is self-rewarding

$$v_{low} = \min_{\substack{(x,y)\in C \\ v\in V}} \left\{ (1-\delta)\, r\,(x,y) + \delta v \right\}$$

$$(1-\delta) r(x,y) + \delta v \geq (1-\delta) r(x,\hat{y}) + \delta v_{low} \text{ all } \hat{y} \in Y$$

  then

$$\Rightarrow v_{low} = (1-\delta) r(h\,(y)\,, y) + \delta v \geq (1-\delta) r(h\,(y)\,, H\,(h\,(y))) + \delta v_{low}$$

- if binds and $v > v_{low}$ then minimize RHS of inequality

$$v_{low} = \min_{y} r(h\,(y)\,, H\,(h\,(y)))$$

Best Value

- for Best, use Worst to punish and Best as reward

  solve:
$$\max_{\substack{(x,y)\in C\\ v\in V}} = \left\{(1-\delta)\,r\,(x,y) + \delta v_{high}\right\}$$
$$(1-\delta)r(x,y) + \delta v_{high} \geq (1-\delta)r(x,\hat{y}) + \delta v_{low} \text{ all } \hat{y} \in Y$$
  then clearly $v_{high} = r\,(x,y)$

- so
$$\max r\,(h\,(y)\,,y)$$
  subject to $r\,(h\,(y)\,,y) \geq (1-\delta)r(h\,(y)\,,H\,(h\,(y))) + \delta v_{low}$

- if constraint not binding $\rightarrow$ Ramsey (first best)

- otherwise value is constrained by v_{low}

Insurance with Limitted Commitment

- 2 agents utility $u\left(c^A\right)$ and $u\left(c^B\right)$
- y_t^A is iid over $[y_{low}, y_{high}]$
- $y_t^B = \bar{y} - y_t^A$ same distribution as y_t^A (symmetry)
- define

$$w_{aut} = \frac{Eu\left(y\right)}{1-\beta}$$

- let $[w_l\left(y\right), w_h\left(y\right)]$ be the set of attainable levels of utility for A when A has income y (by symmetry it is also that of A with income $\bar{y} - y$)
- $v\left(w, y\right)$ for $w \in [w_l, w_h]$ be the highest utility for B given that A is promised w and has income y (the pareto frontier)

Recursive Representation

$$v(w,y) = \max\left\{u\left(c^B\right) + \beta E v\left(w'(y'), y'\right)\right\}$$

$$w = u\left(c^A\right) + \beta E w(y')$$

$$u\left(c^A\right) + \beta E w(y') \geq u(y) + \beta v_{aut}$$

$$u\left(c^B\right) + \beta E v\left(w'(y'), y'\right) \geq u(\bar{y} - y) + \beta v_{aut}$$

$$c^A + c^B \leq \bar{y}$$

$$w'(y') \in [w_l(y'), w_h(y')]$$

- is this a contraction? NO
- is it monotonic? YES
- should solve for $[w_l(y), w_h(y)]$ jointly
  - clearly $w_l(y) = u(y) + \beta v_{aut}$
  - $w_h(y)$ such that $v(w_h(y), y) = u(\bar{y} - y) + \beta v_{aut}$

Stochastic Dynamics

- output of stochastic dynamic programming:

  optimal policy:

$$x_{t+1} = g\left(x_t, z_t\right)$$

- convergence to steady state?

  on rare occasions (but not necessarily never...)

- convergence to something?

Notion of Convergence

Idea:

- start at $t = 0$ with some x_0 and s_0
- compute $x_1 = g(x_0, z_0) \rightarrow x_1$ is not uncertain from $t = 0$ view
- z_1 is realized $\rightarrow$ compute $x_2 = g(x_1, z_1)$

  x_2 is random from point of view of $t = 0$
- continue... x_3, x_4, x_5, ...x_t are random variables from $t = 0$ perspective
- $F_t(x_t)$ distribution of x_t (given x_0, z_0)

  more generally think of joint distribution of (x, z)
- convergence concept

$$\lim_{t \rightarrow \infty} F_t(x) = F(x)$$

Examples

- stochastic growth model
- Brock-Mirman ($\delta = 0$)

$$\begin{aligned} u(c) &= \log c \\ f(A,k) &= Ak^{\alpha} \end{aligned}$$

and A_t is i.i.d. optimal policy

$$k_{t+1} = sA_t k_t^{\alpha}$$

with $s = \beta\alpha$

Examples

- search model: last recitation

  employment state u and e (also wage if we want)

  $\rightarrow$ invariant distribution gives steady state unemployment rate

- if uncertainty is idiosyncratic in a large population

  $\Rightarrow F$ can be interpreted as a cross section

Bewley / Aiyagari

- income fluctuations problem

$$v(a, y; R) = \max_{0 \leq a' \leq Ra+y} \{u(Ra + y - a') + \beta E[v(a', y'; R) | y]\}$$

- solution $a' = g(a, y; R)$
- invariant distribution $F(a; R)$

  cross section assets in large population
- how does F vary with R? (continuously?)
- once we have F can compute moments:

  market clearing

$$\int a dF(a; R) = K$$

Markov Chains

- N states of the world
- let Π_{ij} be probability of $s_{t+1} = j$ conditional on $s_t = i$
- $\Pi = (\Pi_{ij})$ transition matrix
- p distribution over states
- $p_0 \to p_1 = \Pi p_0$ (why?)$\to$ ... $\to$

$$p_t = \Pi^t p_0$$

- does Π^t converge?

Examples

- example 1: Π^t converges
- example 2: transient state
- example 3: Π^t does not converge but flucutates
- example C: ergodic sets

Theorem

Let $S = \{s_1, ...s_l\}$ and Π

a. S can be partitioned into M ergodic sets

b. the sequence

$$\left(\frac{1}{n}\right)\sum_{k=0}^{n-1}\Pi^k \rightarrow Q$$

c. each row of Q is an invariant distribution and so are the convex combinations

Theorem

Let $S = \{s_1, ...s_l\}$ and Π
then Π has a unique ergodic set if and only if there is a state s_j such that for all i there exists an n≥ 1 such that $\pi_{ij}^{(n)} > 0$. In this case Π has a unique invariant distribution p^*; each row of Q equals p^*

Theorem

let $\varepsilon_j^n = \min_i \pi_{ij}^n$ and $\varepsilon^n = \sum_j \varepsilon_j^n$. Then S has a unique ergodic set with no cyclical moving subsets if and only if for some N≥ 1 $\varepsilon^N > 0$. In this case $\Pi^n \to Q$