Recursive Methods

Outline Today's Lecture

- housekeeping: ps#1 and recitation day/ theory general / web page
- finish Principle of Optimality:

  Sequence Problem $\underset{\text{(for values and plans)}}{\Longleftrightarrow}$ solution to Bellman Equation
- begin study of Bellman equation with bounded and continuous F
- tools: contraction mapping and theorem of the maximum

Sequence Problem vs. Functional Equation

- Sequence Problem: (SP)

$$\begin{aligned} V^*(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ & \text{s.t. } x_{t+1} \in \Gamma(x_t) \\ & x_0 \text{ given} \end{aligned}$$

- ... more succinctly

$$V^*(x_0) = \sup_{\tilde{x} \in \Pi(x_0)} u(\tilde{x}) \qquad \text{(SP)}$$

- functional equation (FE) [this particular FE called Bellman Equation]

$$V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \qquad \text{(FE)}$$

Principle of Optimality

IDEA: use BE to find value function V^* and optimal plan x^*

- **Thm 4.2.** V^* defined by SP $\Rightarrow V^*$ solves FE
- **Thm 4.3.** V solves FE and ...... $\Rightarrow V = V^*$
- **Thm 4.4.** $\tilde{x}^* \in \Pi(x_0)$ is optimal

  $\Rightarrow V^*(x_t^*) = F\left(x_t^*, x_{t+1}^*\right) + \beta V^*\left(x_{t+1}^*\right)$
- **Thm 4.5.** $\tilde{x}^* \in \Pi(x_0)$ satisfies $V^*(x_t^*) = F\left(x_t^*, x_{t+1}^*\right) + \beta V^*\left(x_{t+1}^*\right)$ and ......

  $\Rightarrow \tilde{x}^*$ is optimal

Why is this Progress?

- **intuition:** breaks planning horizon into two: ‘now’ and ‘then’
- **notation:** reduces unnecessary notation (especially with uncertainty)
- **analysis:** prove existence, uniqueness and properties of optimal policy (e.g. continuity, montonicity, etc...)
- **computation:** associated numerical algorithm are powerful for many applications

Proof of Theorem 4.3 (max case)

Assume for any $\tilde{x} \in \Pi(x_0)$

$$\lim_{T \to \infty} \beta^T V(x_T) = 0.$$

BE implies

$$\begin{aligned} V(x_0) &\geq F(x_0, x_1) + \beta V(x_1), \text{ all } x_1 \in \Gamma(x_0) \\ &= F(x_0, x_1^*) + \beta V(x_1^*), \text{ some } x_1^* \in \Gamma(x_0) \end{aligned}$$

Substituting $V(x_1)$:

$$\begin{aligned} V(x_0) &\geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 V(x_2), \text{ all } x \in \Pi(x_0) \\ &= F(x_0, x_1^*) + \beta F(x_1^*, x_2^*) + \beta^2 V(x_2^*), \text{ some } x^* \in \Pi(x_0) \end{aligned}$$

Continue this way

$$\begin{aligned} V(x_0) &\geq \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} V(x_{n+1}) \text{ for all } x \in \Pi(x_0) \\ &= \sum_{t=0}^{n} \beta^t F\left(x_t^*, x_{t+1}^*\right) + \beta^{n+1} V\left(x_{n+1}^*\right) \text{ for some } x^* \in \Pi(x_0) \end{aligned}$$

Since $\beta^T V(x_T) \to 0$, taking the limit $n \to \infty$ on both sides of both expressions we conclude that:

$$\begin{aligned} V(x_0) &\geq u(\tilde{x}) \text{ for all } \tilde{x} \in \Pi(x_0) \\ V(x_0) &= u(\tilde{x}^*) \text{ for some } \tilde{x}^* \in \Pi(x_0) \end{aligned}$$

Bellman Equation as a Fixed Point

- define operator

$$T\left(f\right)\left(x\right)=\max_{y\in\Gamma(x)}\left\{F\left(x,y\right)+\beta f\left(y\right)\right\}$$

- V solution of BE $\iff$ V fixed point of T [i.e. $TV=V$]

Bounded Returns:

- if $\|F\|<B$ and F and Γ are continuos: T maps continuous bounded functions into continuous bounded functions
- bounded returns $\Rightarrow$ T is a Contraction Mapping $\Rightarrow$ unique fixed point
- many other bonuses

Needed Tools

- Basic Real Analysis (section 3.1):

  {vector, metric, noSLP, complete} spaces

  cauchy sequences

  closed, compact, bounded sets

- Contraction Mapping Theorem (section 3.2)

- Theorem of the Maximum: study of RHS of Bellman equation (equivalently of T) (section 3.3)