Recursive Methods

Outline Today's Lecture

- finish off: theorem of the maximum
- Bellman equation with bounded and continuous F
- differentiability of value function
- application: neoclassical growth model
- homogenous and unbounded returns, more applications

Our Favorite Metric Space

$$S = \left\{ f : X \to R,\ f \text{ is continuous, and } \|f\| \equiv \sup_{x \in X} |f(x)| < \infty \right\}$$

with

$$\rho(f, g) = \|f - g\| \equiv \sup_{x \in X} |f(x) - g(x)|$$

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

Assume that F is bounded and continuous and that Γ is continuous and has compact range.

Theorem 4.6. T maps the set of continuous and bounded functions S into itself. Moreover T is a contraction.

Proof. That T maps the set of continuous and bounded follow from the Theorem of Maximum (we do this next)

That T is a contraction $\rightarrow$ Blackwell sufficient conditions

$\rightarrow$monotonicity, notice that for $f \geq v$

$$\begin{aligned} Tv(x) &= \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\} \\ &= F(x, g(x)) + \beta v(g(x)) \\ &\leq \{F(x, g(y)) + \beta f(g(x))\} \\ &\leq \max_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\} = Tf(x) \end{aligned}$$

$\rightarrow$discounting: for $a > 0$

$$T(v+a)(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta(v(y) + a)\}$$

$$= \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\} + \beta a = T(v)(x) + \beta a.$$

Theorem of the Maximum

- want T to map continuous function into continuous functions

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

- want to learn about optimal policy of RHS of Bellman

$$G(x) = \arg \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

- First, continuity concepts for correspondences
- ... then, a few example maximizations
- ... finally, Theorem of the Maximum

Continuity Notions for Correspondences

assume Γ is non-empty and compact valued (the set $\Gamma(x)$ is non empty and compact for all $x \in X$)

Upper Hemi Continuity (u.h.c.) at x: for any pair of sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to x$ and $x_n \in \Gamma(y_n)$ there exists a subsequence of $\{y_n\}$ that converges to a point $y \in \Gamma(x)$.

Lower Hemi Continuity (l.h.c.) at x: for any sequence $\{x_n\}$ with $x_n \to x$ and for every $y \in \Gamma(x)$ there exists a sequence $\{y_n\}$ with $x_n \in \Gamma(y_n)$ such that $y_n \to y$.

Continuous at x: if Γ is both upper and lower hemi continuous at x

Max Examples

$$\begin{aligned} h(x) &= \max_{y \in \Gamma(x)} f(x,y) \\ G(x) &= \arg \max_{y \in \Gamma(x)} f(x,y) \end{aligned}$$

ex 1: $f(x,y) = xy$; $X = [-1,1]$; $\Gamma(x) = X$.

$$\begin{aligned} G(x) &= \begin{cases} \{-1\} & x < 0 \\ [-1,1] & x = 0 \\ \{1\} & x > 0 \end{cases} \\ h(x) &= |x| \end{aligned}$$

continued...

ex 2: $f(x,y) = xy^2;\ X = [-1,1];\ \Gamma(x) = X$

$$G(x) = \begin{cases} \{0\} & x < 0 \\ [-1,1] & x = 0 \\ \{-1,1\} & x > 0 \end{cases}$$

$$h(x) = \max\{0, x\}$$

Theorem of the Maximum

Define:

$$\begin{aligned} h(x) &= \max_{y \in \Gamma(x)} f(x, y) \\ G(x) &= \arg \max_{y \in \Gamma(x)} f(x, y) \\ &= \{y \in \Gamma(x) : \; h(x) = f(x, y)\} \end{aligned}$$

Theorem 3.6. (Berge) Let $X \subset \mathbb{R}^l$ and $Y \subset \mathbb{R}^m$. Let $f : X \times Y \rightarrow \mathbb{R}$ be continuous and $\Gamma : X \rightarrow Y$ be compact-valued and continuous. Then $h : X \rightarrow \mathbb{R}$ is continuous and $G : X \rightarrow Y$ is non-empty, compact valued, and u.h.c.

$\lim \max \to \max \lim$

Theorem 3.8. Suppose $\{f_n(x,y)\}$ and $f(x,y)$ are concave in y that and Γ is convex and compact valued.
Then if $f_n \to f$ in the sup-norm (uniformly). Define

$$\begin{aligned} g_n(x) &= \arg\max_{y\in\Gamma(x)} f_n(x,y) \\ g(x) &= \arg\max_{y\in\Gamma(x)} f(x,y) \end{aligned}$$

then $g_n(x) \to g(x)$ for all x (pointwise convergence); if X is compact then the convergence is uniform.

Uses of Corollary of CMThm

Monotonicity of v^*

Theorem 4.7. Assume that $F(\cdot, y)$ is increasing, that Γ is increasing, i.e.

$$\Gamma(x) \subset \Gamma(x')$$

for $x \leq x'$. Then, the unique fixed point v^* satisfying $v^* = Tv^*$ is increasing. If $F(\cdot, y)$ is strictly increasing, so is v^*.

Proof

By the corollary of the CMThm, it suffices to show Tf is increasing if f is increasing. Let $x \leq x'$:

$$\begin{aligned} Tf(x) &= \max_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\} \\ &= F(x,y^*) + \beta f(y^*) \text{ for some } y^* \in \Gamma(x) \\ &\leq F(x',y^*) + \beta f(y^*) \end{aligned}$$

since $y^* \in \Gamma(x) \subset \Gamma(x')$

$$\leq \max_{y \in \Gamma(x')} \{F(x,y) + \beta f(y)\} = Tf(x')$$

If $F(\cdot, y)$ is strictly increasing

$$F(x,y^*) + \beta f(y^*) < F(x',y^*) + \beta f(y^*).$$

Concavity (or strict) concavity of v^*

Theorem 4.8. Assume that X is convex, Γ is concave, i.e. $y \in \Gamma(x)$, $y' \in \Gamma(x')$ implies that

$$y^\theta \equiv \theta y' + (1-\theta) y \in \Gamma(\theta x' + (1-\theta) x) \equiv \Gamma\left(x^\theta\right)$$

for any $x, x' \in X$ and $\theta \in (0,1)$. Finally assume that F is concave in (x, y). Then, the fixed point v^* satisfying $v^* = Tv^*$ is concave in x. Moreover, if $F(\cdot, y)$ is strictly concave, so is v^*.

convergence of policy functions

- with concavity of F and convexity of $\Gamma \to$ optimal policy *correspondence* $G(x)$ is actually a continuous *function* $g(x)$
- since $v_n \to v$ uniformly $\Rightarrow g_n \to g$
  (Theorem 4.8)
- we can use this to derive comparative statics

Differentiability

- can't use same strategy as with monotonicty or concavity: space of differentiable functions is *not* closed

- many envelope theorems, imply differentiability of h

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

- always if formula: if $h(x)$ is differentiable and there exists a $y^* \in int(\Gamma(x))$ then

$$h'(x) = f_x(x, y)$$

  ...but is h differentiable?

continued...

- one approach (e.g. Demand Theory) relies on smoothness of Γ and f (twice differentiability) $\rightarrow$ use f.o.c. and implicit function theorem

- won't work for us since $f(x,y) = F(x,y) + \beta V(y) \rightarrow$ don't know if f is once differentiable yet! $\rightarrow$ going in circles...

Benveniste and Sheinkman

First a Lemma...

Lemma. Suppose $v(x)$ is concave and that there exists $w(x)$ such that $w(x) \leq v(x)$ and $v(x_0) = w(x_0)$ in some neighborhood D of x_0 and w is differentiable at x_0 ($w'(x_0)$ exists) then v is differentiable at x_0 and $v'(x_0) = w'(x_0)$.

Proof. Since v is concave it has at least one subgradient p at x_0:

$$w(x) - w(x_0) \leq v(x) - v(x_0) \leq p \cdot (x - x_0)$$

Thus a subgradient of v is also a subgradient of w. But w has a unique subgradient equal to $w'(x_0)$.

Benveniste and Sheinkman

Now a Theorem

Theorem. Suppose F is strictly concave and Γ is convex. If $x_0 \in int\,(X)$ and $g\,(x_0) \in int\,(\Gamma\,(x_0))$ then the fixed point of T, V, is differentiable at x and

$$V'\,(x) = F_x\,(x, g\,(x))$$

Proof. We know V is concave. Since $x_0 \in int\,(X)$ and $g\,(x_0) \in int\,(\Gamma\,(x_0))$ then $g\,(x_0) \in int\,(\Gamma\,(x))$ for $x \in D$ a neighborhood of x_0 then

$$W\,(x) = F\,(x, g\,(x_0)) + \beta V\,(g\,(x_0))$$

and then $W\,(x) \leq V\,(x)$ and $W\,(x_0) = V\,(x_0)$ and $W'\,(x_0) = F_x\,(x_0, g\,(x_0))$ so the result follows from the lemma.