Recursive Methods

Outline Today's Lecture

- discuss Matlab code
- differentiability of value function
- application: neoclassical growth model
- homogenous and unbounded returns, more applications

Review of Bounded Returns Theorems

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

F is bounded and continuous and Γ is continuous and compact
Theorem 4.6. T is a contraction.
Theorem 4.7. $F(\cdot, y)$ and Γ is increasing $\Rightarrow v^*$ is increasing. If $F(\cdot, y)$ is strictly increasing, $\Rightarrow v^*$ strictly increasing.
Theorem 4.8. $X,\ \Gamma$ convex F is concave in $(x, y) \Rightarrow v^*$is concave in x. If $F(\cdot, y)$ is strictly concave $\Rightarrow v^*$ is strictly concave and the optimal correspondence $G(x)$ is a continuous function $g(x)$.
Theorem 4.9. $g_n \to g$

Differentiability

- can't use same strategy as with monotonicty or concavity: space of differentiable functions is *not* closed
- many envelope theorems, imply differentiability of h

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

- always if formula: if $h(x)$ is differentiable and there exists a $y^* \in int(\Gamma(x))$ then

$$h'(x) = f_x(x, y)$$

  ...but is h differentiable?

continued...

- one approach (e.g. Demand Theory) relies on smoothness of Γ and f (twice differentiability) $\rightarrow$ use f.o.c. and implicit function theorem
- won't work for us since $f(x, y) = F(x, y) + \beta V(y)$ $\rightarrow$ don't know if f is once differentiable yet! $\rightarrow$ going in circles...

Benveniste and Sheinkman

First a Lemma...

Lemma. Suppose $v(x)$ is concave and that there exists $w(x)$ such that $w(x) \leq v(x)$ and $v(x_0) = w(x_0)$ in some neighborhood D of x_0 and w is differentiable at x_0 ($w'(x_0)$ exists) then v is differentiable at x_0 and $v'(x_0) = w'(x_0)$.

Proof. Since v is concave it has at least one subgradient p at x_0:

$$w(x) - w(x_0) \leq v(x) - v(x_0) \leq p \cdot (x - x_0)$$

Thus a subgradient of v is also a subgradient of w. But w has a unique subgradient equal to $w'(x_0)$.

Benveniste and Sheinkman

Now a Theorem

Theorem. Suppose F is strictly concave and Γ is convex. If $x_0 \in int\,(X)$ and $g\,(x_0) \in int\,(\Gamma\,(x_0))$ then the fixed point of T, V, is differentiable at x and

$$V'\,(x) = F_x\,(x, g\,(x))$$

Proof. We know V is concave. Since $x_0 \in int\,(X)$ and $g\,(x_0) \in int\,(\Gamma\,(x_0))$ then $g\,(x_0) \in int\,(\Gamma\,(x))$ for $x \in D$ a neighborhood of x_0 then

$$W\,(x) = F\,(x, g\,(x_0)) + \beta V\,(g\,(x_0))$$

and then $W\,(x) \leq V\,(x)$ and $W\,(x_0) = V\,(x_0)$ and $W'\,(x_0) = F_x\,(x_0, g\,(x_0))$ so the result follows from the lemma