Recursive Methods

Outline Today's Lecture

- neoclassical growth application: use all theorems
- constant returns to scale
- homogenous returns
- unbounded returns

Constant Returns

$$F(\lambda x, \lambda y) = \lambda F(x, y), \text{ for } \lambda \geq 0$$

and,

$$x \in X \implies \lambda x \in X, \text{ for } \lambda \geq 0$$

(i.e. X is a cone)

$$y \in \Gamma(x) \implies \lambda y \in \Gamma(\lambda x), \text{ for } \lambda \geq 0$$

(graph of Γ, A, is a cone)

Restrictions

- since F is unbounded is the $\sup < \infty$? is the max well defined?
- can we apply the Principle of Optimality?

1. restrict Γ: for some α such that $\gamma\beta < 1$:

$$y \in \Gamma(x) \implies \|y\| \leq \alpha \|x\|$$

   "state can't grow too fast"

2. restrict F : for some $0 < B < \infty$

$$|F(x, y)| \leq B(\|x\| + \|y\|) \text{ all } (x, y) \in A$$

   "some weak boundedness condition: only allow unboundedness along rays"

Implications

$$\|x_t\| \leq \alpha^t \|x_0\| \text{ for } x \in \Pi(x_0) \text{ all } x_0 \in X$$

Thus:

$$\begin{aligned}
|u_n(x) - u_{n-1}(x)| &= \beta^t |F(x_t, x_{t+1})| \\
&\leq \beta^t B(\|x_t\| + \|x_{t+1}\|) \\
&= \beta^t B\left(\alpha^t \|x_0\| + \alpha^{t+1} \|x_0\|\right) \\
&= (\beta\alpha)^t B(1+\alpha)\|x_0\| \to 0
\end{aligned}$$

so $u_n(x)$ is Cauchy $\Longrightarrow$ $u_n(x) \to u(x)$
So we have A1 and A2 $\Longrightarrow$ theorems 4.2 and 4.4

supremum's properties

- we established that $v^* : X \to R$
- note that $u(\lambda x) = \lambda u(x)$ and $x \in \Pi(x_0) \implies \lambda x \in \Pi(\lambda x_0)$
- v^* must be homogenous of degree 1

$$\begin{aligned} v^*(\lambda x_0) &= \sup_{x \in \Pi(\lambda x_0)} u(x) \\ &= \sup_{\frac{x}{\lambda} \in \Pi(x_0)} u\left(\lambda \frac{x}{\lambda}\right) \\ &= \lambda \sup_{\tilde{x} \in \Pi(x_0)} u(\tilde{x}) \\ &= \lambda v^*(x_0) \end{aligned}$$

$$
\begin{aligned}
|u(x)| &= \left|\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})\right| \\
&\leq \sum_{t=0}^{\infty} \beta^t \left|F(x_t, x_{t+1})\right| \\
&\leq B \sum_{t=0}^{\infty} \beta^t \left(\alpha^t \|x_0\| + \alpha^{t+1} \|x_0\|\right) \\
&\leq B \|x_0\| \sum_{t=0}^{\infty} (\beta\alpha)^t (1+\alpha) \\
&= \left[B \frac{1+\alpha}{1-\beta\alpha}\right] \|x_0\|
\end{aligned}
$$

$\Longrightarrow$ $|v^*(x)| \leq c \|x_0\|$ for some $c \in R$

What Space to Use?

$$H(X) = \left\{ \begin{array}{c} f : X \to R : f \text{ is continuous and homogenous of degree } 1 \\ \text{and } \frac{f(x)}{\|x\|} \text{ is bounded} \end{array} \right\}$$

$$\|f\| = \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)| = \sup_{x \in X} \frac{|f(x)|}{\|x\|}$$

- $H(X)$ is complete
- define operator $T : H(X) \to H(X)$

$$Tf(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\}$$

Properties

- Operator $T : H(X) \to H(X)$

$$Tf(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\}$$

- note that for any $v \in H(X)$

$$\beta^t |v(x_t)| \leq \beta^t c \|x_t\| \leq (\alpha\beta)^t c \|x_0\| \to 0$$

  thus $\beta^t v(x_t) \to 0$ for all feasible plans (Theorems 4.3 and 4.5 apply)
  $\implies T$ has unique fixed point $v \in H(X)$

- is T is a contraction?

Is T a contraction?

- Modify Blackwell's condition (bounded functions) to show that T it is a contraction; approach in SLP
- Note that

$$\begin{aligned}\frac{Tf}{\|x\|} &= \max_{y\in\Gamma(x)}\left\{\frac{1}{\|x\|}F(x,y)+\beta\frac{1}{\|x\|}f\left(\frac{y}{\|y\|}\|y\|\right)\right\}\\ &= \max_{y\in\Gamma(x)}\left\{F\left(\frac{x}{\|x\|},\frac{y}{\|x\|}\right)+\beta\frac{\|y\|}{\|x\|}f\left(\frac{y}{\|y\|}\right)\right\}\end{aligned}$$

- Idea: study related operator on functions space of continuous functions defined for $\|x\|=1$

Related operator

- Let
$$\tilde{X} = X \cap \{x : \|x\| = 1\}$$

- Define $\tilde{T} : C\left(\tilde{X}\right) \to C\left(\tilde{X}\right)$ as
$$\tilde{T}f = \max_{\substack{y \in \Gamma(x) \\ \|x\|=1}} \left\{ F(x,y) + \beta \|y\| f\left(\frac{y}{\|y\|}\right)\right\}$$

$\tilde{T}$ satifies all our assumptions about bounded returns!

$\implies \tilde{T}$ is a contraction of modulus $\alpha\beta < 1$

Yes, T is a contraction!

- since $\tilde{T}$ is a contraction of modulus $\alpha\beta < 1$

$$\sup_{\tilde{x} \in \tilde{X}} \left| \tilde{T} f - \tilde{T} g \right| \leq \alpha\beta \sup_{\tilde{x} \in \tilde{X}} |f - g|$$

- for $f \in H(X)$

$$\tilde{T} f = \frac{T f}{\|x\|}$$

(note that $f \in H(X)$

- Thus

$$\sup_{x \in X} |T f - T g| = \|x\| \sup_{\tilde{x} \in \tilde{X}} \left| \tilde{T} f - \tilde{T} g \right| \leq \alpha\beta \sup_{\tilde{x} \in \tilde{X}} |f - g| = \alpha\beta \sup_{x \in X} |f - g|$$

so T is a contraction on $H(X)$

Renormalizing

- studying a related operator is convenient in practice
  $\rightarrow$ reduces dimensionality!
- $\|x\| = 1$ not necessarily most convenient normalization ...
- ... another normalization (much used)
  if $x = \left(x^1, x^2\right) \in R^n$ and $x^1 \in R$ then use $x^1 = 1$

Homogenous Returns of Degree θ

similar tricks work (see Alvarez and Stokey, JET)

- rough idea for: $\theta > 0$

$$F(\lambda x, \lambda y) = \lambda^{\theta} F(x, y)$$

$$|F(x, y)| \leq B(\|x\| + \|y\|)^{\theta} \text{ all } (x, y) \in A$$

- Γ as before but now α such that $\gamma \equiv \beta\alpha^{\theta} < 1$
- arguments are exactly parallel
- in particular, T is a contraction of modulus γ
- for $\theta < 0$ and $\theta = 0$ several complications with origin...

  but they can be surmounted

Unbounded Returns and Monotonicity

- numerically cannot handle unbounded returns
- idea: T may *not* be a contraction

  but all is not lost: it still is **monotonic**

Theorem 4.14

1. Start from $v_0 \geq v^*$
2. **IF** $T v_0 = v_1 \leq v_0$ then define $v_n = T^n v_0$ (decreasing sequence)
3. **IF** $\lim_{n\to\infty} v_0 (x_n) \leq 0$ all $x \in \Pi (x_0)$ all x_0

   then clearly $v_n (x) \to v (x)$ for all $x \in X$, for some $v : X \to \bar{R}$
4. **IF** $Tv = v$ (is this implied by $v_n \to v$?)

THEN $v = v^*$

- can be used for quadratic returns

Unbounded Returns and Monotonicity

Squeezing argument:
1. suppose $v_L(x) \leq v^*(x) \leq v^U(x)$
2. and $T^n v^U(x) \to v$ and $T^n v^U(x) \to v$
THEN $v = v^*$

Next Class

- we're done with Chapter 4
- next class: deterministic dynamics
- Chapter 6
- Boldrin-Montruccio 1986 paper