Recursive Methods

Outline Today's Lecture

- "Anything goes": Boldrin Montrucchio
- Global Stability: Liapunov functions
- Linear Dynamics
- Local Stability: Linear Approximation of Euler Equations

Anything Goes

treat $X = [0, 1] \in R$ case for simplicity

- take any $g(x) : [0, 1] \rightarrow [0, 1]$ that is twice continuously differentiable on $[0, 1]$

  $\Rightarrow g'(x)$ and $g''(x)$ exists and are bounded

- define

$$W(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{L}{2}x^2$$

- Lemma: W is strictly concave for large enough L

Proof

$W(x,y) = -\frac{1}{2}y^2 + yg(x) - \frac{L}{2}x^2$

$$\begin{aligned} W_1 &= yg'(x) - Lx \\ W_2 &= -y + g(x) \end{aligned}$$

$$\begin{aligned} W_{11} &= yg''(x) - L \\ W_{22} &= -1 \\ W_{12} &= g'(x) \end{aligned}$$

thus $W_{22} < 0$; $W_{11} < 0$ is satisfied if $L \geq \max_x |g''(x)|$

$$\begin{aligned} W_{11}W_{22} - W_{12}W_{21} &= -yg''(x) + L - g'(x)^2 > 0 \\ &\Rightarrow L > g'(x)^2 + yg''(x) \end{aligned}$$

then $L > [\max_x |g'(x)|]^2 + \max_x |g''(x)|$ will do.

Decomposing W (in a concave way)

- define $V(x) = W(x, g(x))$ and F so that

$$W(x,y) = F(x,y) + \beta V(y)$$

that is $F(x,y) = W(x,y) - \beta V(y)$.

- Lemma: V is strictly concave

Proof: immediate since W is concave and X is convex. Computing the second derivative is useful anyway:

$$V''(x) = g''(x) g(x) + g'(x)^2 - L$$

since $g \in [0,1]$ then clearly our bound on L implies $V''(x) < 0$.

Concavity of F

- Lemma: F is concave for $\beta \in \left[0, \tilde{\beta}\right]$ for some $\tilde{\beta} > 0$

$$\begin{aligned} F_{11}(x,y) &= W_{11}(x,y) = yg''(x) - L \\ F_{12}(x,y) &= W_{12}(x,y) = -1 \\ F_{22}(x,y) &= W_{22} - \beta V_{22} = -1 - \beta\left[g''(x)\,g(x) + g'(x)^2 - L\right] \end{aligned}$$

$$F_{11}F_{22} - F_{12}^2 > 0$$

$$\Rightarrow (yg''(x) - L)\left(-1 - \beta\left[g''(x)\,g(x) + g'(x)^2 - L\right]\right) - g'(x)^2 > 0$$

... concavity of F

- Let

$$\eta_1(\beta) = \min_{x,y}(-F_{22})$$
$$\eta_2(\beta) = \min_{x,y}\left[F_{11}F_{22} - F_{12}^2\right]$$

$$\eta(\beta) = \min\{H_1(\beta), H_2(\beta)\} \geq 0$$

- for $\beta = 0$ $\eta(\beta) > 0$. η is continuous (Theorem of the Maximum) $\Rightarrow$ exists $\tilde{\beta} > 0$ such that $H(\beta) \geq 0$ for all $\beta \in \left[0, \tilde{\beta}\right]$.

Monotonicity

- Use
$$W(x,y) = -\frac{1}{2}y^2 + yg(x) - \frac{L_1}{2}x^2 + L_2 x$$
- L_2 does not affect second derivatives
- claim: F is monotone for large enough L_2