Recursive Methods

Outline Today's Lecture

- linearization argument
- review linear dynamics
- stability theorem for Non-Linear dynamics

Linearization argument

- Euler Equation

$$F_y\left(x, g\left(x\right)\right) + \beta F_x\left(g\left(x\right), g\left(g\left(x\right)\right)\right) = 0$$

- steady state

$$F_y\left(x^*, x^*\right) + \beta F_x\left(x^*, x^*\right) = 0$$

- $g'\left(x^*\right)$ gives dynamics of x_t close to a steady state
- first order Taylor approximation

$$x_{t+1} - x^* \cong g'\left(x^*\right)\left(x_t - x^*\right)$$

- local stability if $\left|g'\left(x^*\right)\right| < 1$

computing $g'(x)$

$$\begin{aligned} 0 &= F_{yx}(x^*, x^*) + F_{yy}(x^*, x^*)\, g'(x^*) + \\ &\quad + \beta F_{xx}(x^*, x^*)\, g'(x^*) + \beta F_{xy}(x^*, x^*) \left[g'(x^*)\right]^2 \end{aligned}$$

- quadratic in $g'(x^*) \Rightarrow$ two candidates for $g'(x^*)$
- reciprocal pairs: λ is a solution so is $1/\lambda\beta$

$$0 = F_{yx}(x^*, x^*) + \left[F_{yy}(x^*, x^*) + \beta F_{xx}(x^*, x^*)\right]\lambda + \beta F_{xy}(x^*, x^*)\,\lambda^2$$

dividing by $\lambda^2\beta$ and since $F_{yx}(x^*, x^*) = F_{xy}(x^*, x^*)$

$$0 = \beta F_{yx}(x^*, x^*) \left[\frac{1}{\lambda\beta}\right]^2 + \left[F_{yy}(x^*, x^*) + \beta F_{xx}(x^*, x^*)\right] \left[\frac{1}{\beta\lambda}\right] + F_{xy}(x^*, x^*)$$

- Thus if $|\lambda_1| < 1 \rightarrow |\lambda_2| > 1$

Using $g'(x^*)$

- x_0 close to the steady state x^*smaller root has absolute value less than one, consider the following sequence of $\{x_{t+1}\}$:

$$x_{t+1} = x^* + g'(x^*)(x_t - x^*) \quad \text{for } t \geq 0$$

- sequence satisfies the Euler Equations
- since $|g'(x^*)| < 1$, it converges to the steady state x^*, and hence it satisfies the transversality condition
- $\Rightarrow$ if F concave we have found a solution
- If both $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then we do not know which one describes $g'(x^*)$ if any, but we do know that that steady state is not locally stable

Neoclassical growth model

$F(x,y) = U(f(x),y)$ so

$$\begin{aligned} F_x(x,y) &= U'(f(x)-y)\,f'(x) \\ F_y(x,y) &= -U'(f(x)-y) \\ F_{xx}(x,y) &= U''(f(x)-y)\,f'(x)^2 + U'(f(x)-y)\,f''(x) \\ F_{yy}(x,y) &= U''(f(x)-y) \\ F_{xy}(x,y) &= -U''(f(x)-y)\,f'(x) \end{aligned}$$

steady state k^* solves $1 = \beta f'(k^*)$

$$\begin{aligned} 0 &= F_{xy} + [F_{yy} + \beta F_{xx}]\,g' + (g')^2 F_{xy} \\ &= -U''f' + \left[U'' + \beta U'' f'^2 + \beta U' f''\right] g' - (g')^2 \beta U'' f' \\ &= -U''\left[1/\beta - \left[1 + 1/\beta + \left(\frac{f''}{f'}/\frac{U''}{U'}\right)\right] g' + (g')^2\right] \end{aligned}$$

quadratic function

$$Q(\lambda) = 1/\beta - \left[1 + 1/\beta + \left(\frac{f''}{f'}/\frac{U''}{U'}\right)\right]\lambda + \lambda^2.$$

Notice that

$$\begin{aligned}
Q(0) &= \frac{1}{\beta} > 0 \\
Q(1) &= -\left(\frac{f''}{f'}/\frac{U''}{U'}\right) < 0 \\
Q'(\lambda^*) &= 0 : 1 < \lambda^* = \left[1 + 1/\beta + \left(\frac{f''}{f'}/\frac{U''}{U'}\right)\right]/2 \\
Q(1/\beta) &= -\left(\frac{f''}{f'}/\frac{U''}{U'}\right)\frac{1}{\beta} < 0 \\
Q(\lambda) &> 0 \text{ for } \lambda \text{ large}
\end{aligned}$$

So,

$$
\begin{aligned}
0 &= Q(\lambda_1) = Q(\lambda_2) \\
0 &< \lambda_1 < 1 < 1/\beta < \lambda_2
\end{aligned}
$$

- smallest root $\lambda_1 = g'(k^*)$ changes with $\frac{f''}{f'} / \frac{U''}{U'}$ controls speed of convergence

Stability of linear dynamic systems of higher dimensions

$$y_{t+1} = Ay_t$$

assume A is non-singular $\rightarrow \bar{y} = 0$

- diagonalizing the matrix A we obtain:

$$A = P^{-1}\Lambda P$$

- Λ is a diagonal matrix with its eigenvalues λ_i on its diagonal
- matrix P contains the eigenvectors of A

continued...

- write linear system as

$$P y_{t+1} = \Lambda \; P y_t \text{ for } t \geq 0$$

- or defining z as $z_t \equiv P y_t$

$$z_{t+1} = \Lambda z_t \text{ for } t \geq 0$$

Stability Theorem

Let λ_i be such that for $i = 1, 2, ..., m$ we have $|\lambda_i| < 1$ and for $i = m+1, m+2, ..., n$ we have $|\lambda_i| \geq 1$. Consider the sequence

$$y_{t+1} = Ay_t \text{ for } t \geq 0$$

for some initial condition y_0. Then

$$\lim_{t \to \infty} y_t = 0,$$

if an only if the initial condition y_0 satisfies:

$$y_0 = P^{-1}\hat{z}_0$$

where $\hat{z}_0$ is a vector with its $n - m$ last coordinates equal to zero, i.e.

$$\hat{z}_{i0} = 0 \text{ for } i = m + 1, m + 2, ..., m$$

Non-Linear version

take $x_{t+1} = h(x_t)$ and let A be the Jacobian $(n \times n)$ of h. Assume $I - A$ is nonsingular. Assume A has eigenvalues λ_i be such that for $i = 1, 2, ..., m$ we have $|\lambda_i| < 1$ and for $i = m+1, m+2, ..., n$ we have $|\lambda_i| \geq 1$. Then there is a neighbourhood of $\bar{x}$, call it U, and a continuously differentiable function $\phi : U \to R^{n-m}$ such that x_t is stable IFF $x_o \in U$ and $\phi(x_0) = 0$. The jacobian of the function ϕ has rank $n - m$.

- idea: can solve ϕ for $n - m$ last coordinates as functions of first m coordinates

Second order differential equation

$$x_{t+2} = A_1 x_{t+1} + A_2 x_t$$

with $x_t \in R^n$ and with initial conditions x_0 and x_{-1}.

- define

$$X_t = \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}$$

- then

$$X_{t+2} = J \; X_t$$

where the matrix $2n \times 2n$ matrix J has four $n \times n$ blocks

$$J = \begin{bmatrix} A_1 & A_2 \\ I & 0 \end{bmatrix}$$

Linearized Euler equations

- Idea: apply second order linear stability theory to linearized Euler
-
$$F_x(x, y) + \beta F_x(y, h(y, x)) = 0$$
- stacked system
$$X_t = \begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix}$$
then $H(X_t) = X_{t+1}$ is
$$H(X_t) = \begin{bmatrix} h(x_{t+1}, x_t) \\ x_{t+1} \end{bmatrix}$$
- then compute the jacobian of H and use our non-linear theorem
- remark: roots will come in reciprocal pairs