Auctions 4:

Multiunit Auctions &

Cremer-McLean Mechanism

M units of the same object are offered for sale.

Each bidder has a set of (marginal values) $V^i = (V_1^i, V_2^i, \ldots V_M^i)$, the objects are substitutes, $V_k^i \geq V_{k+1}^i$.

Extreme cases: unit-demand, the same value for all objects.

- Types of auctions:
- The discriminatory ("pay-your-bid");
- Uniform-price;
- Vickrey;
- Multi-unit English;

- Ausubel;
- Dutch, descending uniform-price,
- ...

Issues: Existence and description of equilibria, price series if sequential, efficiency, optimality, non-homogenous goods, complementarities,...

7 Vickrey Auction

- Let $(b_1^i, b_2^i, \ldots, b_n^i)$ be the vector of bids submitted by i.
- Winners: M highest bids.
- Payments: If player i wins m objects, then has to pay the sum of m highest non-winning bids from the others.

  Or, price for each unit is: minimal value to have and win.

  E.g. to win 3d unit need to bid among $(M-2)$ highest bids, $p = (M-2)$sd highest bid of the others.
- Weakly dominant to bid truthfully, $b_k^i = V_k^i$.

8 Interdependent valuations

8.1 Notation

K objects; given $\mathbf{k} = (k_1, \ldots, k_N)$, denote

$$\mathbf{V^k} = \left(V_1^{k_1}, \ldots, V_N^{k_N}\right).$$

Winners circle at s, $\mathcal{I}^{\mathbf{k}}(\mathrm{s})$, is the set of bidders with the highest value among $\mathbf{V^k}$.

$\mathbf{k}$ is *admissible* if $1 \leq k_i \leq K$ and

$$0 \leq \sum_{i=1}^{N} (k_i - 1) < K.$$

8.2 Single-crossing condition

MSC (*single-crossing*) For any admissible $\mathbf{k}$, for all $\mathbf{x}$ and any pair of players $\{i, j\} \subset \mathcal{I}^{\mathbf{k}}(\mathbf{x})$,

$$\frac{\partial V_i^{k_i}(\mathbf{x})}{\partial x_i} > \frac{\partial V_j^{k_j}(\mathbf{x})}{\partial x_i}.$$

8.3 Efficiency: VCG mechanism (generalized Vickrey auction)

- Allocation rule: Efficient.

- Payments: Vickrey price that player j pays for kth unit won:

$$p_j^k = V_j^k(s_j^k, x_{-j}) =$$

$(M-k+1)$th highest

among $\{V_i^m(s_j^k, x_{-j})\}_{i \neq j}^{m=1..M}$.

These are generically different across units and winners (unlike with private values).

9 Cremer & McLean Mechanism

- Multiple units. Single-crossing and non-independent values.
- Efficient, Extract all the surplus.

Discrete support: $\mathcal{X}^i = \{0, \Delta, 2\Delta, \ldots, (t_i - 1)\Delta\}$, discrete single-crossing is assumed (no need if the values are private).

$\Pi(\mathbf{x})$ is the joint probability of x, $\Pi_i = (\pi(\mathbf{x}_{-i}|x_i))$.

Theorem: In the above conditions and if Π has a full rank, there exists a mechanism in which truth-telling is an efficient ex post equilibrium and in which the seller extracts full surplus from the bidders.

Proof: Consider VCG mechanism $(\mathbf{Q}^*, \mathbf{M}^*)$. Define,

$$U_i^*(x_i) = \sum_{\mathbf{x}_{-i}} \pi(\mathbf{x}_{-i}|x_i) \left[Q_i^*(\mathbf{x}) V_i(\mathbf{x}) - M_i^*(\mathbf{x})\right].$$

This is the expected surplus of buyer i in VCG mechanism. Define, $\mathbf{u}_i^* = (U_i^*(x_i))_{x_i \in \mathcal{X}^i}$.

There exists $\mathbf{c}_i = (c_i(\mathbf{x}_{-i}))_{\mathbf{x}_{-i} \in \mathcal{X}_{-i}}$, such that $\Pi_i \mathbf{c}_i = \mathbf{u}_i^*$. Equivalently,

$$\sum_{\mathbf{x}_{-i}} \pi(\mathbf{x}_{-i}|x_i) c_i(\mathbf{x}_{-i}) = U_i^*(x_i).$$

Then, *CM* mechanism $(\mathbf{Q}^*, \mathbf{M}^{CM})$ is defined by

$$M_i^{\mathbf{CM}}(\mathbf{x}) = M_i^*(\mathbf{x}) + c_i(\mathbf{x}_{-i}).$$

Remarks:

- Private values (correlated), equiv. second price auction with additional payments.

- Negative payoffs sometimes, not ex post IR, payoffs arbitrarily large if the distribution converges to the independent one.

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