Psychology and Economics (Lecture 24)

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1 Animal metabolism and other “universal” laws

- How much energy E does one need to make an animal of size M function?

- The naive guess would be CRS:

$$E \sim M$$

  If the size of the animal doubles, one needs twice the amount of energy.

- But Nature does better than that.

- If the size of the animal double, one needs less than twice the amount of energy.

$$E \sim M^{3/4}$$
$$\ln E = a + \frac{3}{4} \ln M$$

- Explanation: West, Brown, Enquist (*Science* '97, *Nature* '99).

- Lots of "universal" laws similar in biology, physics. Understanding them is a hot area of research.

- In physics, "universality" has a precise, technical meaning: after rescaling, different metals etc. behave exactly the same.

- Perhaps there should be in economics:
  - Zipf's law $P\,(\text{Size} > x) \sim x^{-1}$: Cities, Firms (Axtell, *Science* '01)
  - Power law in the stock market: 3 for returns and number of trades, 3/2 for volumes. Theory in Gabaix et al. (*Nature* '03).

2 Zipf's law

- That's the statement that $\zeta = 1$.

- Original Zipf's law: Frequency of words in a text: Estoup (1916), Zipf (1949)

- Original power law in economics: For incomes, Pareto (1897)

- Zipf's law holds for cities

- Take U.S. Order cities by size. Largest: NYC $= \#1$, LA $=\#2$,...

- Regression with the 135 largest American metropolitan areas 1991.

$$\ln \text{Rank} = 10.53 - \underset{(.010)}{1.005} \ln \text{Size} \tag{1}$$

- $R^2 = .986$. (No tautology) "One of the strongest [non-trivial] facts in social sciences".

- This means $\ln P(\text{Size} > S) = a - \zeta \ln S$ with $\zeta \simeq 1$, or

$$P(\text{Size} > S) \sim S^{-\zeta} \tag{2}$$

- If largest city has 10 million inhabitants, the 10th city has 1 million, the 100th city 100,000... + interpretation with ratios

- One first explanation: Monkeys at a typewriter (Mandelbrot, 1961). Exercise: Work it out.

3 Power laws in Economics

$$F(x) = P(S > x) \simeq \frac{k}{x^{\zeta}}$$
$$f(x) = -F'(x) = \frac{k\zeta}{x^{\zeta+1}}$$

- Empirically, we see:

$$\ln P(S > x) \simeq -\zeta \ln x + c$$
$$\ln f(x) \simeq -(\zeta + 1) \ln x + c'$$

3.1 Simplest way to estimate ζ

- Rank units by size: $S_{(1)} \geq S_{(2)} \geq \ldots$

- Plot lnSize vs lnRank

- See above which size one has a straight line

- In that domain, run:

$$\ln \text{Rank} = -\zeta \ln \text{Size} + C$$

- True standard error: $\widehat{\zeta} 2/n$.

4 Axtell (2001)

- Studies the 5 million firms in the US, in 1997.

$$\ln f(S) = a - (\zeta + 1) \ln S$$
$$R^2 = 0.993$$

$$\zeta = 1.059 \simeq 1$$

That's Zipf's law: $\zeta = 1$.

5 Other domains for Zipf's law

- Firms, size of bankruptcy, number of workers in strikes, exports

- Assets under management of mutual funds

- Popularity (number of clicks) of internet sites: Huberman (1999), Barabasi and Albert (1999)

- $\zeta =$ "power law exponent" $=$ "Pareto exponent"

- Low ζ means high inequality.

- “Universal” laws: Same laws in different countries, time period, economic structures, trading mechanisms.

- $\rightarrow$Need simple explanations that do not depend on the details of the system. Ideally, we want no tunable parameters.

- Other quantitative “laws” in economics? (besides Black-Sholes): Quantity theory of money $PY \sim M$.

6 Lots of power laws in physics

- Similar power laws are found in: earthquakes, solar eruptions, extinction of species, pieces of a vase.

- No general theory explains them and they do not have a "mathematical " exponent like 1.

- Reference: Didier Sornette, "Critical Phenomena in Natural Sciences" (Springer , 2003)

- Work often done by / with physicists ("econophysics", $\sim$ 150 physicists working on this). More empirical that "Sante Fe" research, "complexity theory".

7 An explanation. Zipf's law with exponent 1

- Start from an arbitrary initial distribution.

- Cities follow similar processes: e.g. grow at 2%/year, ±.5%. ("Gibrat's law").

- Consequence: the distribution converges to a steady state distribution which is Zipf, with exponent 1.

- The explanation is robust to new cities, several regions with different means and variances.

- Gibrat's law appears to be true empirically.

8 From Gibrat's law to Zipf's law (Gabaix '99)

- S_t^i =(Size of city i at time t) / (Total expected population at time t).

$$E\left[\sum_{i=1}^{N} S_t^i\right] = 1 \tag{3}$$

- Evolution

$$S_{t+1}^i = \gamma_{t+1}^i S_t^i \tag{4}$$

where γ_{t+1}^i =normalized growth rate of city i. i.i.d. and independent of i, with probability density $f(\gamma)$.

- (3) and (4) imply: $E[\gamma_{t+1}^i] = 1$, or

$$\int_0^\infty \gamma f(\gamma) d\gamma = 1 \tag{5}$$

- Distribution: $G_t(S) = P(S_t > S)$ has equation of motion:

$$\begin{aligned} G_{t+1}(S) &= P(S_{t+1} > S) = P(\gamma_{t+1} S_t > S) = E[1_{S_t > S/\gamma_{t+1}}] \\ &= E[E[1_{S_t > S/\gamma_{t+1}} | \gamma_{t+1}]] = E[G_t(S/\gamma_{t+1})] \\ &= \int_0^\infty G_t(S/\gamma) f(\gamma) d\gamma \end{aligned} \tag{6}$$

- Suppose that there is a steady state $G_t = G$. It verifies

$$G(S) = \int_0^\infty G(S/\gamma) f(\gamma) d\gamma \tag{7}$$

- Try the solution $G(S) = a/S$ (=Zipf's law), using (5):

$$\begin{aligned}\int_0^\infty G(S/\gamma)f(\gamma)d\gamma &= \int_0^\infty a/(S/\gamma)f(\gamma)d\gamma \\ &= a/S \cdot \int_0^\infty \gamma f(g)d\gamma = a/S \\ &= G(S)\end{aligned}$$

- So $G(S) = a/S$ satisfies the equation (7) of steady-state process: It works.

- In fact (Theorem), there is a steady state distribution, and it must be a/S.