14.13 Economics and Psychology (Lecture 5)

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1 Second order risk aversion for EU

- The agent takes the 50/50 gamble $\Pi + \sigma$, $\Pi - \sigma$ iff:

$$B(\Pi) = \frac{1}{2}u(x + \sigma + \Pi) + \frac{1}{2}u(x - \sigma + \Pi) \geq u(x)$$

i.e. $\Pi \geq \Pi^*$ where:

$$B(\Pi^*) = u(x)$$

- Assume that u is twice differentiable and take a look at the Taylor expansion of the above equality for small σ. .

$$B(\Pi) = u(x) + \frac{1}{2}u'(x)\,2\Pi + \frac{1}{4}u''(x)\,2\left[\sigma^2 + \Pi^2\right] + o\left(\sigma^2 + \Pi^2\right) = u(x)$$

then

$$\Pi = \frac{\rho}{2}\left[\sigma^2 + \Pi^2\right] + o\left(\sigma^2 + \Pi^2\right)$$

where $\rho = -\frac{u''}{u'}$

- To solve : $\Pi = \frac{\rho}{2}\left[\sigma^2 + \Pi^2\right]$ for small σ. Call $\rho' = \rho/2$.

- *Barbarian way*: Solve:

$$\Pi^2 - \frac{1}{\rho'}\Pi + \sigma^2 = 0$$

Exactly. Then take Taylor. One finds:

$$\Pi = \rho'\sigma^2 = \frac{\rho}{2}\sigma^2$$

- Elegant way: $\Pi = \rho' \left[\sigma^2 + \Pi^2\right]$ for small σ.

  - Π will be small. Take a guess. If the expansion is $\Pi = k\sigma$, then we get:

$$k\sigma = \rho' \left[\sigma^2 + k^2\sigma^2\right]$$
$$k = \rho'\sigma \left[1 + k^2\right]$$

    contraction for $\sigma \to 0$, the RHS goes to 0 and the LHS is k. This guess doesn't work.

  - Let's try instead $\Pi = k\sigma^2$. Then:

$$k\sigma^2 = \rho' \left[\sigma^2 + k^2\sigma^4\right]$$
$$= \rho'\sigma^2 + o\left(\sigma^2\right)$$

$$\Rightarrow k = \rho' + o(1) \text{ after dividing both side by } \sigma^2$$

that works, with $k = \rho'$. Conclusion:

$$\Pi = \frac{\rho}{2}\sigma^2.$$

- Note this method is really useful when the equation to solve doesn't have a closed form solution. For example, solve for small σ

$$\pi = \rho'(\sigma^2 + \pi^2 + \pi^7)$$

solution postulate $\Pi = k\sigma^2$, plug it back in the equation to solve, then take $\sigma \longrightarrow 0$ and it works for $k = \rho'$

- The σ^2 indicates "second order" risk aversion.

2 First order risk aversion of PT

- Consider same gamble as for EU. Take the gamble iff $\Pi \geq \Pi^*$ where

$$\pi(.5)u(\Pi^* + \sigma) + \pi(.5)u(\Pi^* - \sigma) = 0$$

- We will show that in PT, as $\sigma \to 0$, the risk premium Π is of the order of σ when reference wealth $x = 0$. This is called the *first order risk aversion*.

- Let's compute Π for $u(x) = x^\alpha$ for $x \geq 0$ and $u(x) = -\lambda |x|^\alpha$ for $x \leq 0$.

- The premium Π at $x = 0$ satisfies

$$0 = \pi(\frac{1}{2})\,(\sigma + \Pi^*)^{\alpha} + \pi(\frac{1}{2})\,(-\lambda)\,|-\sigma + \Pi^*|^{\alpha}$$

cancel $\pi(\frac{1}{2})$ and use the fact that $-\sigma + \Pi^* < 0$ to get

$$\begin{aligned} 0 &= (\sigma + \Pi^*)^{\alpha} - \lambda(\sigma - \Pi^*)^{\alpha} \\ &\iff (\sigma + \Pi^*)^{\alpha} = \lambda(\sigma - \Pi^*)^{\alpha} \\ &\iff \sigma + \Pi^* = \lambda^{1/\alpha}\,[\sigma - \Pi^*] \end{aligned}$$

then

$$\Pi^* = \frac{\lambda^{\frac{1}{\alpha}} - 1}{\lambda^{\frac{1}{\alpha}} + 1}\sigma = k\sigma$$

where k is defined appropriately.

- Empirically:

$$\lambda = 2,\ \alpha \simeq 1$$
$$k \simeq \frac{2-1}{2+1} = \frac{1}{3}$$

- Note that when $\lambda = 1$, the agent is risk neutral and the risk premium is 0.

2.1 Calibration 1

- Consider an EU agent with a constant elasticity of substitution, CES, utility, i.e. $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

- **Gamble 1**
  \$50,000 with probability 1/2
  \$100,000 with probability 1/2

- **Gamble 2.** $\$x$ for sure.

- Typical x that makes people indifferent between the two gambles belongs to $(60k, 75k)$ (though some people are risk loving and ask for higher x).

- If $x = 65k$, what is γ

$$.5\ u(W+50) + .5\ u(W+100) = u(W+x)$$
$$.5 \cdot W^{1-\gamma} \cdot 50^{1-\gamma} + .5 \cdot W^{1-\gamma} \cdot 100^{1-\gamma} = W^{1-\gamma} \cdot x^{1-\gamma}$$
$$5 \cdot 50^{1-\gamma} + .5 \cdot 100^{1-\gamma} = x^{1-\gamma}$$

- Note the relation between x and the elasticity of substitution γ:

x	$75k$	$70k$	$63k$	$58k$	$54k$	$51.9k$	$51.2k$
γ	0	1	3	5	10	20	30

Right γ seems to be between 1 and 10.

- Evidence on financial markets calls for γ bigger than 10. This is the equity premium puzzle.

2.2 Calibration 2

- **Gamble 1**
  \$11 with probability 1/2
  \$-10 with probability 1/2

- **Gamble 2.** Get \$0 for sure.

- If someone prefers Gamble 2, she or he satisfies
$$u(W) > \frac{1}{2}u(W + \Pi - \sigma) + \frac{1}{2}u(W + \Pi + \sigma).$$
Here, $\Pi = \$.5$ and $\sigma = \$10.5$. We know that in EU
$$\Pi < \Pi^* = \frac{\rho}{2}\sigma^2$$

And thus with CES utility $\rho = -\frac{u''(W)}{u'(W)} = -\frac{-\gamma W^{-\gamma-1}}{W^{-\gamma}} = \frac{\gamma}{W}$

$$\Pi < \frac{\rho}{2}\sigma^2 = \frac{\gamma}{2W}\sigma^2 \Leftrightarrow \frac{2W\Pi}{\sigma^2} < \gamma$$

forces large γ as the wealth W is larger than 10^5 easily.

- Here:

$$\gamma > \frac{2W\Pi}{\sigma^2} = \frac{2 \cdot 10^5 \cdot .5}{10.5^2} \approx 10^3$$

- Conclusion: very hard to calibrate the same model to large and small gambles using EU.

2.3 Calibration Conclusions

- What would a PT agent do? If $\alpha = 1$, $\lambda = 2$, in calibration 2 he won't take gamble 1 as

$$\pi(.5)11^{\alpha} + \pi(.5)(-\lambda \cdot 10^{\alpha}) = -9\pi(.5) < 0$$

- In PT we have $\Pi^* = k\sigma$. For $W = 10^4$, $\gamma = 2$, and $\sigma = 0.5$ the risk premium is $\Pi^* = k\sigma = \frac{1}{3} \cdot .5 \approx \$.2$ while in EU $\Pi^* = \frac{\gamma}{2W}\sigma^2 \approx \$.00002$

- If we want to fit an EU parameter γ to a PT agent we get

$$\Pi^{PT}(\sigma) = \Pi^{EU}(\sigma)$$
$$k\sigma = \frac{\gamma}{2W}\sigma^2$$

then

$$\hat{\gamma} = \frac{2kW}{\sigma}$$

and this explodes as $\sigma \longrightarrow 0$.

- If someone is averse to 50-50 lose \$100/gain g for all wealth levels then he or she will turn down 50-50 lose L-gain G in the table

- Guess:

$L\backslash g$	\$101	\$105	\$110	\$125
\$400	\$400	\$420	\$550	
\$800	\$800			
\$1000	\$1,010			
\$2000				
\$10,000				

$L \backslash g$	\$101	\$105	\$110	\$125
\$400	\$400	\$420	\$550	\$1,250
\$800	\$800	\$1,050	\$2,090	∞
\$1000	\$1,010	\$1,570	∞	∞
\$2000	\$2,320	∞	∞	∞
\$10,000	∞	∞	∞	∞

cf paper by Matt Rabin

2.4 What does it mean?

- EU is still good for modelling.
- Even behavioral economists stick to it when they are not interested in risk taking behavior, but in fairness for example.
- The reason is that EU is nice, simple, and parsimonious.

3 Two extensions of PT

- Both outcomes, x and y, are positive, $0 < y < x$. Then,

$$V = v(y) + \pi(p)(v(x) - v(y)).$$

  Why not $V = \pi(p)v(x) + \pi(1-p)v(y)$? Because it becomes self-contradictory when $x = y$ and we stick to K-T calibration that puts $\pi(.5) < .5$.

- Continuous gambles, distribution $f(x)$

  EU gives:

$$V = \int_{-\infty}^{+\infty} u(x) f(x) \, dx$$

  PT gives:

$$\begin{aligned} V = & \int_{0}^{+\infty} u(x) f(x) \pi' (P(g \geq x)) \, dx \\ & + \int_{-\infty}^{0} u(x) f(x) \pi' (P(g \leq x)) \, dx \end{aligned}$$