Problem 1. *(Multitype Erdös- Rényi)*
Consider the following multitype generalization of the Erdös-Rényi random graph model.

- Nodes are of two types, type a and type b.
- Fraction f of the nodes are type a, that is we have fn nodes of type a and $(1-f)n$ nodes of type b.
- For each pair of nodes i, j , the edge (i, j) is present, independent from the presence or absence of other edges, with probability p_{ij} given by:

$$p_{i,j} = \begin{cases} \frac{\lambda_a}{n} & \text{if both } i, j \text{ are of type } a. \\ \frac{\lambda_b}{n} & \text{if both } i, j \text{ are of type } b. \\ \frac{\lambda_0}{n} & \text{if } i \text{ and } j \text{ are of different types.} \end{cases}$$

What is the degree distribution of an agent of type a? What is the average degree distribution? Can you find a threshold for a k node complete graph to emerge in population a but not in population b ? Explain.

Problem 2. *(Navigating a Structured Network on a Hypercube)*
Consider a network in which each individual has a binary code represented by a vector of length K. An individual with a given code is linked to all other individuals with exactly same code and also to those who have codes that differ by one entry. Thus, the network can be thought as a hypercube or a regular lattice. So, for instance if $K = 4$, then an individual with code $(0, 1, 0, 0)$ is connected to individuals of the same type as well as individuals of types $(0, 0, 0, 0)$, $(1, 1, 0, 0)$, $(0, 1, 1, 0)$ and $(0, 1, 0, 1)$. Assume that there are m individuals with each code. Calculate the average distance in (shortest) path length between two nodes picked uniformly at random (allowing for the second node to be identical to the first). How does the average distance vary with the number of individuals in the society $n = m2^K$? How does average degree grow in this network? Compare the average degree and distance of this model and the small world model discussed in class (2-D grid model for $\alpha = 2$).

Problem 3. *(Sequential Duel)*
In a sequential duel, two people alternatively have the opportunity to shoot each other; each has an infinite supply of bullets. On each of her turns, a person may shoot or refrain from doing so. Each of person i's shots hits (and kills) its target with probability p_i (independent of whether any other shots hit their targets). Each person cares about her probability of survival (not about the other person's survival). Model this situation as an extensive game with perfect information and chance moves. Show that the strategy pairs in which neither person ever shoots and in which each person always shoots are both subgame perfect equilibria (note that the game does not have a finite horizon, so backward induction cannot be used).

Problem 4. *(Choosing the estate tax rate as an extensive form game)*
Senate is to choose the estate tax rate $x \in X = \{0, 0.01, 0.02, ..., 0.99, 1\}$. There are 45 hard-core Republicans, represented by the Majority Leader, 40 hard-core Democrats, represented by the Minority Leader, and 14 Moderates, represented by the Moderate Leader. The payoff of Republicans is $1 - x$; the payoff of Democrats is x, and the payoff of Moderates is x if $x \leq 1/2$ and $1 - x$ if $x \geq 1/2$. The current estate tax rate is $x_0 = 0.6$ (i.e., 60%).
First, the Majority Leader introduces a bill $x_1 \in X$. Then, the Minority Leader introduces an amendment $x_2 \in X$. According to the Senate rules, first the amendment x_2 is voted against the bill x_1 and the winner of these two is voted against x_0. The winner of the last vote is introduced as the estate tax rate. The winner in each vote is the alternative that collects 50 or more votes. In each vote, first the Majority Leader votes for all the hard-core Republicans, then the Minority Leader votes for the hard-core Democrats, and finally the Moderate Leader votes for the Moderates. Use backwards induction to compute an equilibrium of this

game. [You can restrict your attention to the case $x_1 \leq 0.5 \leq x_2$; you dont need to describe entire strategy, but you need to determine the outcomes of the votes, x_2 as a function of x_1, and x_1.]

Problem 5. Consider three parties choosing their location on an ideological space represented by $[0,1]$. Voters are uniformly distributed over $[0,1]$ and each voter will vote for the closest party. The parties maximize their vote share (the share of total voters voting for them). Show that there does not exist a pure strategy equilibrium.

Problem 6. *(The Colonel Blotto Game)*
Two armies are fighting a war. There are three battlefields. Each army consists of six units. The armies must each decide how many units to place on each battlefield. They do this without knowing the number of units that the other army has committed to a given battlefield. The army who has the most units on a given battlefield wins that battle, and the army that wins the most battles wins the war. If the armies each have the same number of units on a given battlefield, then there is an equal chance that either army wins that battle. A pure strategy for an army is a list (u_1, u_2, u_3) of the number of units it places on battlefields 1, 2, and 3, respectively, where each u_k is in $0, 1, \ldots, 6$ and the sum of the u_k s is 6. For example, if army A allocates its units $(3,2,1)$, and army B allocates its units $(0,3,3)$, then army A wins the first battle, army B wins the second and third battles, and army B wins the war.

1. Argue that there is no pure strategy Nash equilibrium in this game.
2. Show that mixing uniformly at random over all possible configurations of units is not a mixed strategy Nash equilibrium. (Hint: placing all units on one battlefield is not a good idea).
3. Show that each army mixing with equal probability between $(0,3,3)$, $(3,0,3)$, and $(3,3,0)$ is not an equilibrium.