6.207/14.15: Networks
Lecture 10: Introduction to Game Theory—2

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Outline

- Review
- Examples of Pure Strategy Nash Equilibria
- Mixed Strategies
- Existence of Mixed Strategy Nash Equilibrium in Finite Games
- Characterizing Mixed Strategy Equilibria
- Applications

- Reading:
- Osborne, Chapters 3-5.

Pure Strategy Nash Equilibrium

Definition

(Nash equilibrium) *A (pure strategy) Nash Equilibrium of a strategic game* $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *is a strategy profile* $s^* \in S$ *such that for all* $i \in \mathcal{I}$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

- Why is this a "reasonable" notion?
- No player can profitably deviate given the strategies of the other players. Thus in Nash equilibrium, "best response correspondences intersect".
- Put differently, the conjectures of the players are *consistent*: each player i chooses s_i^* expecting all other players to choose s_{-i}^*, and each player's conjecture is verified in a Nash equilibrium.

Examples: Bertrand Competition

- An alternative to the Cournot model is the Bertrand model of oligopoly competition.
- In the Cournot model, firms choose quantities. In practice, choosing prices may be more reasonable.
- What happens if two producers of a homogeneous good charge different prices? *Reasonable answer:* everybody will purchase from the lower price firm.
- In this light, suppose that the demand function of the industry is given by $Q(p)$ (so that at price p, consumers will purchase a total of $Q(p)$ units).
- Suppose that two firms compete in this industry and they both have marginal cost equal to $c > 0$ (and can produce as many units as they wish at that marginal costs).

Bertrand Competition (continued)

- Then the profit function of firm i can be written as

$$\pi_i\left(p_i, p_{-i}\right) = \begin{cases} Q\left(p_i\right)\left(p_i - c\right) & \text{if } p_{-i} > p_i \\ \frac{1}{2}Q\left(p_i\right)\left(p_i - c\right) & \text{if } p_{-i} = p_i \\ 0 & \text{if } p_{-i} < p_i \end{cases}$$

- Actually, the middle row is arbitrary, given by some ad hoc "tiebreaking" rule. Imposing such tie-breaking rules is often not "kosher" as the homework will show.

Proposition

In the two-player Bertrand game there exists a unique Nash equilibrium given by $p_1 = p_2 = c$.

Bertrand Competition (continued)

Proof: Method of "finding a profitable deviation".

- Can $p_1 \geq c > p_2$ be a Nash equilibrium? No because firm 2 is losing money and can increase profits by raising its price.
- Can $p_1 = p_2 > c$ be a Nash equilibrium? No because either firm would have a profitable deviation, which would be to reduce their price by some small amount (from p_1 to $p_1 - \varepsilon$).
- Can $p_1 > p_2 > c$ be a Nash equilibrium? No because firm 1 would have a profitable deviation, to reduce its price to $p_2 - \varepsilon$.
- Can $p_1 > p_2 = c$ be a Nash equilibrium? No because firm 2 would have a profitable deviation, to increase its price to $p_1 - \varepsilon$.
- Can $p_1 = p_2 = c$ be a Nash equilibrium? Yes, because no profitable deviations. Both firms are making zero profits, and any deviation would lead to negative or zero profits.

Examples: Second Price Auction

- Second Price Auction (with Complete Information) The second price auction game is specified as follows:
- An object to be assigned to a player in $\{1, .., n\}$.
- Each player has her own valuation of the object. Player i's valuation of the object is denoted v_i. We further assume that $v_1 > v_2 > ... > 0$.
- Note that for now, we assume that everybody knows all the valuations $v_1, \ldots, v_n$, i.e., this is a complete information game. We will analyze the incomplete information version of this game in later lectures.
    - The assignment process is described as follows:
    - The players simultaneously submit bids, $b_1, .., b_n$.
    - The object is given to the player with the highest bid (or to a random player among the ones bidding the highest value).
    - The winner pays the second highest bid.
    - The utility function for each of the players is as follows: the winner receives her valuation of the object minus the price she pays, i.e., $v_i - b_j$; everyone else receives 0.

Second Price Auction (continued)

Proposition

In the second price auction, truthful bidding, i.e., $b_i = v_i$ for all i, is a Nash equilibrium.

Proof: We want to show that the strategy profile $(b_1, .., b_n) = (v_1, .., v_n)$ is a Nash Equilibrium—**a truthful equilibrium**.

- First note that if indeed everyone plays according to that strategy, then player 1 receives the object and pays a price v_2.
- This means that her payoff will be $v_1 - v_2 > 0$, and all other payoffs will be 0. Now, player 1 has no incentive to deviate, since her utility can only decrease.
- Likewise, for all other players $v_i \neq v_1$, it is the case that in order for v_i to change her payoff from 0 she needs to bid more than v_1, in which case her payoff will be $v_i - v_1 < 0$.
- Thus no incentive to deviate from for any player.

Second Price Auction (continued)

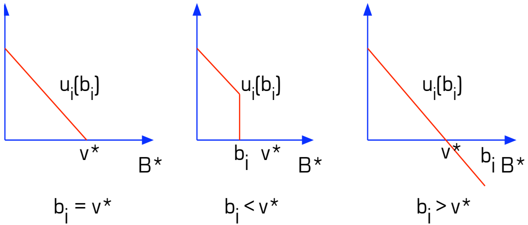
- Are There Other Nash Equilibria? In fact, there are also unreasonable Nash equilibria in second price auctions.
- We show that the strategy $(v_1, 0, 0, \ldots, 0)$ is also a Nash Equilibrium.
- As before, player 1 will receive the object, and will have a payoff of $v_1 - 0 = v_1$. Using the same argument as before we conclude that none of the players have an incentive to deviate, and the strategy is thus a Nash Equilibrium.
- It can be verified the strategy $(v_2, v_1, 0, 0, \ldots, 0)$ is also a Nash Equilibrium.
- Why?

Second Price Auction (continued)

- Nevertheless, the truthful equilibrium, where , $b_i = v_i$, is the **Weakly Dominant Nash Equilibrium**
- In particular, truthful bidding, $b_i = v_i$, weakly dominates all other strategies.
- Consider the following picture proof where B^* represents the maximum of all bids excluding player i's bid, i.e.

$$B^* = \max_{j \neq i} b_j,$$

and v^* is player i's valuation and the vertical axis is utility.



Second Price Auction (continued)

- The first graph shows the payoff for bidding one's valuation. In the second graph, which represents the case when a player bids lower than their valuation, notice that whenever $b_i \leq B^* \leq v^*$, player i receives utility 0 because she loses the auction to whoever bid B^*.
- If she would have bid her valuation, she would have positive utility in this region (as depicted in the first graph).
- Similar analysis is made for the case when a player bids more than their valuation.

- An immediate implication of this analysis is that other equilibria involve the play of weakly dominated strategies.

Nonexistence of Pure Strategy Nash Equilibria

- Example: Matching Pennies.

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- No pure Nash equilibrium.
- How would you play this game?

Nonexistence of Pure Strategy Nash Equilibria

- Example: The Penalty Kick Game.

penalty taker \ goalie	left	right
left	$(-1, 1)$	$(1, -1)$
right	$(1, -1)$	$(-1, 1)$

- No pure Nash equilibrium.
- How would you play this game if you were the penalty taker?
  - Suppose you always show up left.
  - Would this be a "good strategy"?
- Empirical and experimental evidence suggests that most penalty takers "randomize" $\longrightarrow$ mixed strategies.

Mixed Strategies

- Let Σ_i denote the set of probability measures over the pure strategy (action) set S_i.
  - For example, if there are two actions, S_i can be thought of simply as a number between 0 and 1, designating the probability that the first action will be played.
- We use $\sigma_i \in \Sigma_i$ to denote the mixed strategy of player i, and $\sigma \in \Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$ to denote a mixed strategy profile.
- Note that this implicitly assumes that **players randomize independently**.
- We similarly define $\sigma_{-i} \in \Sigma_{-i} = \prod_{j \neq i} \Sigma_j$.
- Following von Neumann-Morgenstern expected utility theory, we extend the payoff functions u_i from S to Σ by

$$u_i(\sigma) = \int_S u_i(s) d\sigma(s).$$

Mixed Strategy Nash Equilibrium

Definition

(Mixed Nash Equilibrium): A mixed strategy profile σ^* is a (mixed strategy) Nash Equilibrium if for each player i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \qquad \text{for all } \sigma_i \in \Sigma_i.$$

Proposition

Let $G = \langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ be a finite strategic form game. Then, $\sigma^* \in \Sigma$ is a Nash equilibrium if and only if for each player $i \in \mathcal{I}$, every pure strategy in the support of σ_i^* is a best response to σ_{-i}^*.

Proof idea: If a mixed strategy profile is putting positive probability on a strategy that is not a best response, then shifting that probability to other strategies would improve expected utility.

Mixed Strategy Nash Equilibria (continued)

- It follows that **every action** in the support of any player's equilibrium mixed strategy yields the same payoff.
- **Implication:** it is sufficient to check pure strategy deviations, i.e., σ^* is a mixed Nash equilibrium if and only if for all i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

- **Note:** this characterization result extends to **infinite games**: $\sigma^* \in \Sigma$ is a Nash equilibrium if and only if for each player $i \in \mathcal{I}$, no action in S_i yields, given σ_{-i}^*, a payoff that exceeds his equilibrium payoff, the set of actions that yields, given σ_{-i}^*, a payoff less than his equilibrium payoff has σ_i^*-measure zero.

Examples

Example: Matching Pennies.

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- Unique mixed strategy equilibrium where both players randomize with probability $1/2$ on heads.

Example: Battle of the Sexes Game.

Player 1 \ Player 2	ballet	football
ballet	$(1, 4)$	$(0, 0)$
football	$(0, 0)$	$(4, 1)$

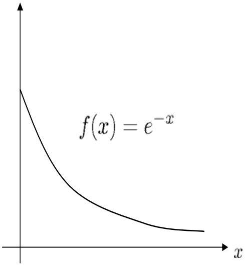
- This game has two pure Nash equilibria and a mixed Nash equilibrium $\left((\frac{4}{5}, \frac{1}{5}), (\frac{1}{5}, \frac{4}{5})\right)$.

Weierstrass's Theorem

Theorem

(Weierstrass) *Let A be a nonempty compact subset of a finite dimensional Euclidean space and let $f : A \longrightarrow \mathbb{R}$ be a continuous function. Then there exists an optimal solution to the optimization problem*

$$\begin{array}{ll} \textit{minimize} & f(x) \\ \textit{subject to} & x \in A. \end{array}$$



$\min_{x \geq 0} \ e^{-x} = 0$

There exists no optimal x that attains it

Kakutani's Fixed Point Theorem

Theorem

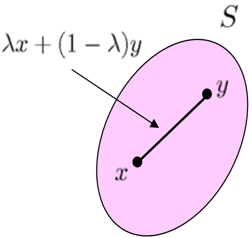
(Kakutani) *Let $f : A \rightrightarrows A$ be a correspondence, with $x \in A \mapsto f(x) \subset A$, satisfying the following conditions:*

- *A is a compact, convex, and non-empty subset of a finite dimensional Euclidean space.*
- *$f(x)$ is non-empty for all $x \in A$.*
- *$f(x)$ is a convex-valued correspondence: for all $x \in A$, $f(x)$ is a convex set.*
- *$f(x)$ has a closed graph: that is, if $\{x^n, y^n\} \to \{x, y\}$ with $y^n \in f(x^n)$, then $y \in f(x)$.*

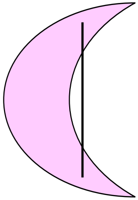
Then, f has a fixed point, that is, there exists some $x \in A$, such that $x \in f(x)$.

Definitions (continued)

- A set in a Euclidean space is compact if and only if it is bounded and closed.
- A set S is **convex** if for any $x, y \in S$ and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda) y \in S$.

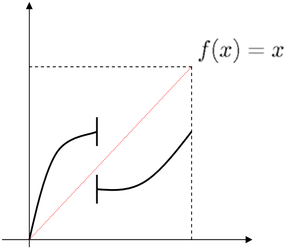


convex set

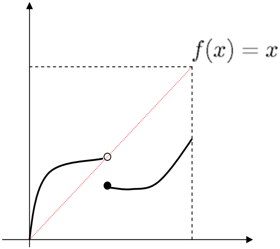


not a convex set

Kakutani's Fixed Point Theorem—Graphical Illustration



$f(x)$ is not convex-valued



$f(x)$ does not have a closed graph

Nash's Theorem

Theorem

(Nash) *Every finite game has a mixed strategy Nash equilibrium.*

- Implication: matching pennies necessarily has a mixed strategy equilibrium.
- Why is this important?
  - Without knowing the existence of an equilibrium, it is difficult (perhaps meaningless) to try to understand its properties.
  - Armed with this theorem, we also know that every finite game has an equilibrium, and thus we can simply try to locate the equilibria.

Proof

- Recall that σ^* is a (mixed strategy) Nash Equilibrium if for each player i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \qquad \text{for all } \sigma_i \in \Sigma_i.$$

- Define the best response correspondence for player i $B_i : \Sigma_{-i} \rightrightarrows \Sigma_i$ as

$$B_i(\sigma_{-i}) = \left\{\sigma_i' \in \Sigma_i \mid u_i(\sigma_i', \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}) \text{ for all } \sigma_i \in \Sigma_i\right\}.$$

- Define the set of best response correspondences as

$$B(\sigma) = [B_i(\sigma_{-i})]_{i \in \mathcal{I}}.$$

- Clearly

$$B : \Sigma \rightrightarrows \Sigma.$$

Proof (continued)

- The idea is to apply Kakutani's theorem to the best response correspondence $B : \Sigma \rightrightarrows \Sigma$. We show that $B(\sigma)$ satisfies the conditions of Kakutani's theorem.

1. Σ is compact, convex, and non-empty.
   - By definition
   $$\Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$$
   where each $\Sigma_i = \{x \mid \sum x_i = 1\}$ is a *simplex* of dimension $|S_i| - 1$, thus each Σ_i is closed and bounded, and thus compact. Their finite product is also compact.
2. $B(\sigma)$ is non-empty.
   - By definition,
   $$B_i(\sigma_{-i}) = \arg\max_{x \in \Sigma_i} u_i(x, \sigma_{-i})$$
   where Σ_i is non-empty and compact, and u_i is linear in x. Hence, u_i is continuous, and by Weirstrass's theorem $B(\sigma)$ is non-empty.

Proof (continued)

3. $B(\sigma)$ is a convex-valued correspondence.
   - Equivalently, $B(\sigma) \subset \Sigma$ is convex if and only if $B_i(\sigma_{-i})$ is convex for all i. Let $\sigma_i', \sigma_i'' \in B_i(\sigma_{-i})$.
   - Then, for all $\lambda \in [0,1] \in B_i(\sigma_{-i})$, we have

$$u_i(\sigma_i', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for all } \tau_i \in \Sigma_i,$$

$$u_i(\sigma_i'', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for all } \tau_i \in \Sigma_i.$$

   - The preceding relations imply that for all $\lambda \in [0,1]$, we have

$$\lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda) u_i(\sigma_i'', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for all } \tau_i \in \Sigma_i.$$

   By the linearity of u_i,

$$u_i(\lambda \sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \text{for all } \tau_i \in \Sigma_i.$$

   Therefore, $\lambda \sigma_i' + (1-\lambda)\sigma_i'' \in B_i(\sigma_{-i})$, showing that $B(\sigma)$ is convex-valued.

Proof (continued)

4. $B(\sigma)$ has a closed graph.
   - Supposed to obtain a contradiction, that $B(\sigma)$ does not have a closed graph.
   - Then, there exists a sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in B(\sigma^n)$, but $\hat{\sigma} \notin B(\sigma)$, i.e., there exists some i such that $\hat{\sigma}_i \notin B_i(\sigma_{-i})$.
   - This implies that there exists some $\sigma'_i \in \Sigma_i$ and some $\epsilon > 0$ such that

$$u_i(\sigma'_i, \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon.$$

   - By the continuity of u_i and the fact that $\sigma^n_{-i} \to \sigma_{-i}$, we have for sufficiently large n,

$$u_i(\sigma'_i, \sigma^n_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) - \epsilon.$$

Proof (continued)

- [step 4 continued] Combining the preceding two relations, we obtain

$$u_i(\sigma'_i, \sigma^n_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon \geq u_i(\hat{\sigma}^n_i, \sigma^n_{-i}) + \epsilon,$$

  where the second relation follows from the continuity of u_i. This contradicts the assumption that $\hat{\sigma}^n_i \in B_i(\sigma^n_{-i})$, and completes the proof.

- The existence of the fixed point then follows from Kakutani's theorem.
- If $\sigma^* \in B(\sigma^*)$, then by definition σ^* is a mixed strategy equilibrium.

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