Cooperative Games

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Coalitional Games

A coalitional (or cooperative) game is a model that focuses on the behavior of groups of players. The strategic interaction is not explicitly modeled as in the case of non-cooperative games.

- N: finite set of `players`
- a `coalition` is any group of players, $S \subseteq N$ (N is the `grand coalition`)
- $v(S) \geq 0$: `worth` of coalition S
- S can divide $v(S)$ among its members; S may implement any payoffs $(x_i)_{i \in S}$ with $\sum_{i \in S} x_i = v(S)$ (no externalities)
- `outcome`: a `partition` $(S_k)_{k=1,\ldots,\bar{k}}$ of N and an `allocation` $(x_i)_{i \in N}$ specifying the division of the worth of each S_k among its members,

$$S_j \cap S_k = \emptyset, \forall j \neq k \ \& \ \bigcup_{k=1}^{\bar{k}} S_k = N$$

$$\sum_{i \in S_k} x_i = v(S_k), \forall k \in \{1, \ldots, \bar{k}\}$$

Examples

A majority game

- Three parties (players 1,2, and 3) can share a unit of total surplus.
- Any majority—coalition of 2 or 3 parties—may control the allocation of output.
- Output is shared among the members of the winning coalition.

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$$

Firm and workers

- A firm, player 0, may hire from the pool of workers $\{1, 2, \ldots, n\}$.
- Profit from hiring k workers is $f(k)$.

$$v(S) = \begin{cases} f(|S| - 1) & \text{if } 0 \in S \\ 0 & \text{otherwise} \end{cases}$$

The Core

Suppose that it is efficient for the grand coalition to form:

$$v(N) \geq \sum_{k=1}^{\bar{k}} v(S_k) \text{ for every partition } (S_k)_{k=1,\ldots,\bar{k}} \text{ of } N.$$

Which allocations $(x_i)_{i \in N}$ can the grand coalition choose? No coalition S should want to break away from $(x_i)_{i \in N}$ and implement a division of $v(S)$ that all its members prefer to $(x_i)_{i \in N}$.

For an allocation $(x_i)_{i \in N}$, use notation $x_S = \sum_{i \in S} x_i$. Allocation $(x_i)_{i \in N}$ is `feasible` for the grand coalition if $x_N = v(N)$.

Definition 1

Coalition S can `block` the allocation $(x_i)_{i \in N}$ if $x_S < v(S)$. An allocation is in the `core` of the game if (1) it is feasible for the grand coalition; and (2) it cannot be blocked by any coalition. $\mathcal{C}$ denotes the set of core allocations,

$$\mathcal{C} = \left\{(x_i)_{i \in N} \middle| x_N = v(N) \;\&\; x_S \geq v(S), \forall S \subseteq N\right\}.$$

Examples

- Two players split \$1, with outside options p and q

$$v(\{1\}) = p, v(\{2\}) = q, v(\{1,2\}) = 1$$
$$\mathcal{C} = \{(x_1, x_2) | x_1 + x_2 = 1, x_1 \geq p, x_2 \geq q\}$$

  What happens for $p = q = 0$? What if $p + q > 1$?
- The majority game

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$$
$$\mathcal{C} = ?$$

- A set A of 1000 sellers interacts with a set B of 1001 buyers in a market for an indivisible good. Each seller supplies one unit of the good and has reservation value 0. Every buyer demands a single unit and has reservation price 1.

$$v(S) = \min(|S \cap A|, |S \cap B|)$$
$$\mathcal{C} = ?$$

Balancedness

Which games have nonempty core?

A vector $(\lambda_S \geq 0)_{S \subseteq N}$ is `balanced` if

$$\sum_{\{S \subseteq N | i \in S\}} \lambda_S = 1, \forall i \in N.$$

A payoff function v is `balanced` if

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N) \text{ for every balanced } \lambda.$$

Interpretation: each player has a unit of time, which can be distributed among his coalitions. If each member of coalition S is active in S for λ_S time, a payoff of $\lambda_S v(S)$ is generated. A game is `balanced` if there is no allocation of time across coalitions that yields a total value $> v(N)$.

Balancedness is Necessary for a Nonempty Core

Suppose that $\mathcal{C} \neq \emptyset$ and consider $x \in \mathcal{C}$. If $(\lambda_S)_{S \subseteq N}$ is balanced, then

$$\sum_{S \subseteq N} \lambda_S v(S) \leq \sum_{S \subseteq N} \lambda_S x_S = \sum_{i \in N} x_i \sum_{S \ni i} \lambda_S = \sum_{i \in N} x_i = v(N).$$

Hence v is balanced.

Balancedness turns out to be also a sufficient condition for the non-emptiness of the core...

Nonempty Core

Theorem 1 (Bondareva 1963; Shapley 1967)

A coalitional game has non-empty core iff it is balanced.

Proof

Consider the linear program

$$X := \min \sum_{i \in N} x_i$$
$$\text{s.t.} \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N.$$

$\mathcal{C} \neq \emptyset \iff X \leq v(N)$ (1)

Dual program

$$Y := \max \sum_{S \subseteq N} \lambda_S v(S)$$
$$\text{s.t.} \quad \lambda_S \geq 0, \forall S \subseteq N \ \& \sum_{S \ni i} \lambda_S = 1, \forall i \in N.$$

v is balanced $\iff Y \leq v(N)$ (2)

The primal linear program has an optimal solution. By the duality theorem of linear programming, $X = Y$ (3).

(1)-(3): $\mathcal{C} \neq \emptyset \iff v$ is balanced

Simple Sufficient Condition for Nonempty Cores

Definition 2

A game v is convex if for any pair of coalitions S and T,

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

Convexity implies that the marginal contribution of a player i to a coalition increases as the coalition expands,

$$S \subset T \,\&\, i \notin T \implies v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Indeed, if v is convex then

$$v((S \cup \{i\}) \cup T) + v((S \cup \{i\}) \cap T) \geq v(S \cup \{i\}) + v(T),$$

which can be rewritten as

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Convex Games Have Nonempty Cores

Theorem 2

Every convex game has a non-empty core.

Define the allocation x with $x_i = v(\{1,\ldots,i\}) - v(\{1,\ldots,i-1\})$. Prove that $x \in \mathcal{C}$. For all $i_1 < i_2 < \cdots < i_k$,

$$\begin{aligned}
\sum_{j=1}^{k} x_{i_j} &= \sum_{j=1}^{k} v(\{1,\ldots,i_j-1,i_j\}) - v(\{1,\ldots,i_j-1\}) \\
&\geq \sum_{j=1}^{k} v(\{i_1,\ldots,i_{j-1},i_j\}) - v(\{i_1,\ldots,i_{j-1}\}) \\
&= v(\{i_1,i_2,\ldots,i_k\}),
\end{aligned}$$

where the inequality follows from $\{i_1,\ldots,i_{j-1}\} \subseteq \{1,\ldots,i_j-1\}$ and v's convexity.

Core Tâtonnement

Consider a game v with $\mathcal{C} \neq \emptyset$.

- $e(S;x) = v(S) - x_S$: `excess` of coalition S at allocation x
- $D(x) \subseteq 2^N$: `most discontent` coalitions at x,

$$D(x) = \arg\max_{S \in N} w(S)e(S;x)$$

where $w : 2^N \to (0, \infty)$ describes coalitions' relative ability of expressing discontent and threatening to block

For any feasible allocation x^0, consider the following recursive process. For $t = 1, 2, \ldots$

- if $x^{t-1} \in \mathcal{C}$, then $x^t = x^{t-1}$;
- otherwise, one coalition $S^{t-1} \in D(x^{t-1})$ most discontent with x^{t-1} is chosen and $e(S^{t-1}; x^{t-1})$ is transferred symmetrically from $N \setminus S^{t-1}$ to S^{t-1},

$$x_i^t = \begin{cases} x_i^{t-1} + \frac{e(S^{t-1};x^{t-1})}{|S^{t-1}|} & \text{if } i \in S^{t-1} \\ x_i^{t-1} - \frac{e(S^{t-1};x^{t-1})}{|N \setminus S^{t-1}|} & \text{if } i \in N \setminus S^{t-1} \end{cases}.$$

Core Convergence Result

Theorem 3

The sequence (x^t) converges to a core allocation.

For intuition, view allocations x as elements of $\mathbb{R}^N$.

- (x^t) is confined to the hyperplane $\{x|x_N = v(N)\}$.
- Assume that (x^t) does not enter C.
- At each step t, the reallocation is done such that x^{t+1} is the projection of x^t on the hyperplane F_{S^t}, where $F_S = \{x|x_S = v(S)\ \&\ x_N = v(N)\}$.
- Distance from x^t to F_{S^t} is proportional to $e(S^t; x^t)$.
- For any fixed $c \in C$, since x^t and c are on different sides of the hyperplane F_{S^t} and the line $x^t x^{t+1}$ is perpendicular to F_{S^t}, we have $x^t \widehat{x^{t+1}} c > \pi/2$ and $d(x^t, c) \geq d(x^{t+1}, c)$ for all $t \geq 0$.
- $l_c := \lim_{t\to\infty} d(x^t, c)$

Continuation of Proof Sketch

- For any limit point x of (x_t), there exists a subsequence of (x_t) converging to x and a coalition S such that $S^t = S$ along the subsequence.
- The projection of the subsequence on F_S converges to the projection y of x on $S \Rightarrow y$ is also a limit point.
- If $x \notin F_S$ $(x \neq y)$, then for any $c \in \mathcal{C}$ the segment xc is longer than yc because $\widehat{xyc} > \pi/2$. This contradicts $d(x,c) = d(y,c) = l_c$.
- Therefore, $x \in F_S$ and $e(S;x) = 0$. Then $x \in \mathcal{C}$ since, by continuity, S is one of the most discontent coalitions under $x \Rightarrow l_x = 0$.
- Any other limit point z satisfies $d(z,x) = l_x = 0$, so $z = x$.
- (x_t) converges to $x \in \mathcal{C}$.

Singleton Solution Concepts

Two players split \$1, with outside options p and q

$$v(\{1\}) = p, v(\{2\}) = q, v(\{1,2\}) = 1$$
$$\mathcal{C} = \{(x_1, x_2) | x_1 + x_2 = 1, x_1 \geq p, x_2 \geq q\}$$

What happens for $p = q = 0$? What if $p + q > 1$?

The core may be empty or quite large, which compromises its role as a predictive theory. Ideally, select a unique outcome for every cooperative game.

A `value` for cooperative games is a function from the space of games (N, v) to feasible allocations x ($x_N = v(N)$).

The Shapley Value

Shapley (1953) proposed a solution with many economically desirable and mathematically elegant properties.

Definition 3

The `Shapley value` of a game with worth function v is given by

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)).$$

Interpretation: players are randomly ordered in a line, all orders being equally likely. $\varphi_i(v)$ represents the expected value of player i's contribution to the coalition formed by the players preceding him in line.

Why do values sum to $v(N)$?

What's the Shapley value in the divide the dollar game?

Proposition 2$\Rightarrow$ for `convex` games v, $\varphi(v)$ is a convex combination of core allocations. Since $\mathcal{C}$ is convex, $\varphi(v) \in \mathcal{C}$. Not true in general.

Axioms

What is special about the Shapley value?

Axiom 1 (Symmetry)

Players i and j are `interchangeable` *in v if* $v(S \cup \{i\}) = v(S \cup \{j\})$ *for all S disjoint from* $\{i, j\}$*. If i and j are interchangeable in v then* $\varphi_i(v) = \varphi_j(v)$*.*

Axiom 2 (Dummy Player)

Player i is a `dummy` *in v if* $v(S \cup \{i\}) = v(S)$ *for all S. If i is a dummy in v then* $\varphi_i(v) = 0$*.*

Axiom 3 (Additivity)

For any two games v and w, we have $\varphi(v + w) = \varphi(v) + \varphi(w)$*.*

Theorem 4

A value satisfies the three axioms iff it is the Shapley value.

Proof of "If" Part

The only axiom not checked immediately is symmetry. Suppose that i and j are interchangeable. Then

$$\begin{aligned}
\varphi_i(v) &= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}(v(S \cup \{i\}) - v(S)) \\
&= \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}(v(S \cup \{i\}) - v(S)) \\
&+ \sum_{S \subseteq N \setminus \{i,j\}} \frac{(|S|+1)!(|N|-(|S|+1)-1)!}{|N|!}(v(S \cup \{i,j\}) - v(S \cup \{j\})) \\
&= \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}(v(S \cup \{j\}) - v(S)) \\
&+ \sum_{S \subseteq N \setminus \{i,j\}} \frac{(|S|+1)!(|N|-(|S|+1)-1)!}{|N|!}(v(S \cup \{i,j\}) - v(S \cup \{i\})) \\
&= \varphi_j(v).
\end{aligned}$$

Proof of “Only If” Part

Suppose that ψ satisfies the three axioms. We argue that $\psi = \varphi$.

For any non-empty coalition T, define the game

$$v^T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise} \end{cases}.$$

Fix $a \in \mathbb{R}$. By the *symmetry* axiom, $\psi_i(av^T) = \psi_j(av^T)$ for all $i, j \in T$. By the *dummy player* axiom, $\psi_i(av^T) = 0$ for all $i \notin T$. Hence

$$\psi_i(av^T) = \begin{cases} a/|T| & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases},$$

so $\psi(av^T) = \varphi(av^T)$.

Proof of "Only If" Part

The $(2^{|N|} - 1)$ games v^T span the linear space of all games. If we view games as $(2^{|N|} - 1)$-dimensional vectors, it is sufficient to show that the vectors corresponding to the games (v^T) are linearly independent.

For a contradiction, suppose that $\sum_{T \subseteq N} \alpha^T v^T = 0$ with not all α's equal to zero. Let S be a set with minimal cardinality satisfying $\alpha^S \neq 0$. Then $\sum_{T \subseteq N} \alpha^T v^T(S) = \alpha^S \neq 0$, a contradiction.

Thus any v can be written as $v = \sum_{T \subseteq N} \alpha^T v^T$. The additivity of ψ and φ imply

$$\psi(v) = \psi\left(\sum_{T \subseteq N} \alpha^T v^T\right) = \sum_{T \subseteq N} \psi(\alpha^T v^T) = \sum_{T \subseteq N} \varphi(\alpha^T v^T) = \varphi\left(\sum_{T \subseteq N} \alpha^T v^T\right) = \varphi(v).$$

An Alternative Characterization

Equity requirement: for any pair of players, the amounts that each player gains or loses from the other's withdrawal from the game are equal. For a game (N, v), we denote by $v|M$ its restriction to the players in M.

Definition 4

A value ψ has `balanced contributions` if for every game (N, v) we have

$$\psi_i(v|N) - \psi_i(v|N \setminus \{j\}) = \psi_j(v|N) - \psi_j(v|N \setminus \{i\}), \forall i, j \in N.$$

Theorem 5

The unique value that has balanced contributions is the Shapley value.

Proof

At most one value has balanced contributions.

- For a contradiction, let φ' and φ'' be two different such values.
- Let (N, v) be a game with minimal $|N|$ for which the two values yield different outcomes.
- Then for all $i, j \in N$, $\varphi'_i(v|N \setminus \{j\}) = \varphi''_i(v|N \setminus \{j\})$ and $\varphi'_j(v|N \setminus \{i\}) = \varphi''_j(v|N \setminus \{i\})$, along with the balancedness of φ' and φ'', imply $\varphi'_i(v|N) - \varphi''_i(v|N) = \varphi'_j(v|N) - \varphi''_j(v|N)$.
- Since $\sum_{i \in N}(\varphi'_i(v|N) - \varphi''_i(v|N)) = 0$, we obtain $\varphi'_i(v|N) - \varphi''_i(v|N) = 0, \forall i \in N$, or $\varphi'(v|N) = \varphi''(v|N)$, a contradiction.

Next argue that the Shapley value has balanced contributions.

- The Shapley value φ is a linear function of the game, so the set of games for which φ satisfies balanced contributions is closed under linear combinations.
- Since any game is a linear combination of games v^T, it is sufficient to show that these games satisfy balanced contributions...

The Bargaining Problem

The *non-cooperative approach* involves explicitly modeling the bargaining process as an extensive form game (e.g., Rubinstein's (1982) alternating offer bargaining model).

The *axiomatic approach* abstracts away from the details of the bargaining process. Determine directly "reasonable" or "natural" properties that outcomes should satisfy.

What are "reasonable" axioms?

- Consider a situation where two players must split $1. If no agreement is reached, then the players receive nothing.
- If preferences over monetary prizes are identical, then we expect that each player obtains 50 cents.
- Desirable properties: efficiency and symmetry of the allocation for identical preferences.

Nash Bargaining Solution

A `bargaining problem` is a pair (U, d) where $U \subset \mathbb{R}^2$ and $d \in U$.

- U is convex and compact
- there exists some $u \in U$ such that $u > d$

Denote the set of all possible bargaining problems by $\mathcal{B}$. A `bargaining solution` is a function $f : \mathcal{B} \to \mathbb{R}^2$ with $f(U, d) \in U$.

Definition 5

The `Nash (1950) bargaining solution` f^N is defined by

$$\{f^N(U, d)\} = \arg\max_{u \in U, u \geq d} (u_1 - d_1)(u_2 - d_2).$$

Given the assumptions on (U, d), the solution to the optimization problem *exists* and is *unique*.

Axioms

Axiom 4 (Pareto Efficiency)

A bargaining solution f is `Pareto efficient` *if for any bargaining problem* (U, d)*, there does not exist* $(u_1, u_2) \in U$ *such that* $u_1 \geq f_1(U, d)$ *and* $u_2 \geq f_2(U, d)$*, with at least one strict inequality.*

Axiom 5 (Symmetry)

A bargaining solution f is `symmetric` *if for any symmetric bargaining problem* (U, d) *(*$(u_1, u_2) \in U$ *if and only if* $(u_2, u_1) \in U$ *and* $d_1 = d_2$*), we have* $f_1(U, d) = f_2(U, d)$*.*

Axioms

Axiom 6 (Invariance to Linear Transformations)

A bargaining solution f is `invariant` if for any bargaining problem (U, d) and all $\alpha_i \in (0, \infty), \beta_i \in \mathbb{R}$ ($i = 1, 2$), if we consider the bargaining problem (U', d') with

$$\begin{aligned} U' &= \{(\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2) \mid (u_1, u_2) \in U\} \\ d' &= (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2) \end{aligned}$$

then $f_i(U', d') = \alpha_i f_i(U, d) + \beta_i$ for $i = 1, 2$.

Axiom 7 (Independence of Irrelevant Alternatives)

A bargaining solution f is `independent` if for any two bargaining problems (U, d) and (U', d) with $U' \subseteq U$ and $f(U, d) \in U'$, we have $f(U', d) = f(U, d)$.

Characterization

Theorem 6

f^N is the unique bargaining solution that satisfies the four axioms.

Check that f^N satisfies the axioms.

1. Pareto efficiency: follows from the fact that $(u_1 - d_1)(u_2 - d_2)$ is increasing in u_1 and u_2.
2. Symmetry: if (U, d) is a symmetric bargaining problem then $(f_2^N(U, d), f_1^N(U, d)) \in U$ also solves the optimization problem. By the uniqueness of the optimal solution, $f_1^N(U, d) = f_2^N(U, d)$.
3. Independence of irrelevant alternatives: if $f^N(U, d) \in U' \subseteq U$. The value of the objective function for (U', d) cannot exceed that for (U, d). Since $f^N(U, d) \in U'$, the two values must be equal, and by the uniqueness of the optimal solution, $f^N(U, d) = f^N(U', d)$.
4. Invariance to linear transformations: $f^N(U', d')$ is an optimal solution for

$$\max_{\{(u_1', u_2') | u_1' = \alpha_1 u_1 + \beta_1, u_2' = \alpha_2 u_2 + \beta_2, (u_1, u_2) \in U\}} (u_1' - \alpha_1 d_1 - \beta_1)(u_2' - \alpha_2 d_2 - \beta_2) \ldots$$

Proof

Show that for any f that satisfies the axioms, $f(U, d) = f^N(U, d), \forall (U, d)$.

Fix a bargaining problem (U, d) and let $z = f^N(U, d)$. There exists $\alpha_i > 0, \beta_i$ such that the transformation $u_i \to \alpha_i u_i + \beta_i$ takes d_i to 0 and z_i to $1/2$. Define

$$U' = \{(\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2) | (u_1, u_2) \in U\}.$$

Since f and f^N satisfy the invariance to linear transformations axiom, $f(U, d) = f^N(U, d)$ iff $f(U', 0) = f^N(U', 0) = (1/2, 1/2)$. It suffices to prove $f(U', 0) = (1/2, 1/2)$.

Proof

The line $\{(u_1, u_2)|u_1 + u_2 = 1\}$ is tangent to the hyperbola $\{(u_1, u_2)|u_1 u_2 = 1/4\}$ at the point $(1/2, 1/2)$. Given that $f^N(U', 0) = (1/2, 1/2)$, argue that $u_1 + u_2 \leq 1$ for all $u \in U'$.

Since U' is bounded, we can find a rectangle U'' with one side along the line $u_1 + u_2 = 1$, symmetric with respect to the line $u_1 = u_2$, such that $U' \subseteq U''$ and $(1/2, 1/2)$ is on the boundary of U''. Since f is efficient and symmetric, it must be that $f(U'', 0) = (1/2, 1/2)$.

f satisfies independence of irrelevant alternatives, so $f(U'', 0) = (1/2, 1/2) \in U' \subseteq U'' \Rightarrow f(U', 0) = (1/2, 1/2)$

Bargaining with Alternating Offers

- players $i = 1, 2$; $j = 3 - i$
- set of feasible utility pairs

$$U = \{(u_1, u_2) \in [0, \infty)^2 | u_2 \leq g_2(u_1)\}$$

- g_2 s. decreasing, concave, $g_2(0) > 0$
- disagreement point $d = (0, 0)$
- δ_i: discount factor of player i
- at every time $t = 0, 1, \ldots$, player $i(t)$ proposes an alternative $u = (u_1, u_2) \in U$ to player $j(t) = 3 - i(t)$

$$i(t) = \begin{cases} 1 \text{ for} & t \text{ even} \\ 2 \text{ for} & t \text{ odd} \end{cases}$$

  - if $j(t)$ accepts the offer, game ends yielding payoffs $(\delta_1^t u_1, \delta_2^t u_2)$
  - otherwise, game proceeds to period $t + 1$

Subgame perfect equilibrium

Define $g_1 = g_2^{-1}$. Graphs of g_2 and g_1^{-1}: Pareto-frontier of U

Let (m_1, m_2) be the unique solution to the following system of equations

$$\begin{aligned} m_1 &= \delta_1 g_1 (m_2) \\ m_2 &= \delta_2 g_2 (m_1). \end{aligned}$$

(m_1, m_2) is the intersection of the graphs of $\delta_2 g_2$ and $(\delta_1 g_1)^{-1}$.

Subgame perfect equilibrium in "stationary" strategies: in any period where player i has to make an offer to j, he offers u with $u_j = m_j$ and $u_i = g_i(m_j)$, and j accepts only offers u with $u_j \geq m_j$.

Nash Bargaining

Assume g_2 is decreasing, s. concave and continuously differentiable.

Nash bargaining solution:

$$\{u^*\} = \arg\max_{u \in U} u_1 u_2.$$

Theorem 7 (Binmore, Rubinstein and Wolinsky 1985)

Suppose that $\delta_1 = \delta_2 =: \delta$ in the alternating bargaining model. Then the unique SPE payoffs converge to the Nash bargaining solution as $\delta \to 1$.

$$m_1 g_2(m_1) = m_2 g_1(m_2)$$

$(m_1, g_2(m_1))$ and $(g_1(m_2), m_2)$ belong to the intersection of g_2's graph with the same hyperbola, which approaches the hyperbola tangent to the boundary of U (at u^*) as $\delta \to 1$.