Single-Deviation Principle and Bargaining

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Multi-stage games with observable actions

- finite set of players N
- stages $t = 0, 1, 2, \ldots$
- H: set of terminal histories (sequences of action profiles of possibly different lengths)
- at stage t, after having observed a non-terminal history of play $h_t = (a^0, \ldots, a^{t-1}) \notin H$, each player i simultaneously chooses an action $a_i^t \in A_i(h_t)$
- $u_i(h)$: payoff of $i \in N$ for terminal history $h \in H$
- σ_i: behavior strategy for $i \in N$ specifies $\sigma_i(h) \in \Delta(A_i(h))$ for $h \notin H$

Often natural to identify “stages” with time periods.

Examples

- repeated games
- alternating bargaining game

Unimprovable Strategies

To verify that a strategy profile σ constitutes a subgame perfect equilibrium (SPE) in a multi-stage game with observed actions, it suffices to check whether there are any histories h_t where some player i can gain by deviating from playing $\sigma_i(h_t)$ at t and conforming to σ_i elsewhere.

$u_i(\sigma|h_t)$: expected payoff of player i in the subgame starting at h_t and played according to σ thereafter

Definition 1

A strategy σ_i is *unimprovable* given σ_{-i} if $u_i(\sigma_i, \sigma_{-i}| h_t) \geq u_i(\sigma'_i, \sigma_{-i}| h_t)$ for every h_t and σ'_i with $\sigma'_i(h) = \sigma_i(h)$ for all $h \neq h_t$.

Continuity at Infinity

If σ is an SPE then σ_i is unimprovable given σ_{-i}. For the converse...

Definition 2

A game is *continuous at infinity* if

$$\lim_{t\to\infty} \sup_{\{(h,\tilde{h})|h_t=\tilde{h}_t\}} |u_i(h) - u_i(\tilde{h})| = 0, \forall i \in N.$$

Events in the distant future are relatively unimportant.

Single (or One-Shot) Deviation Principle

Theorem 1

Consider a multi-stage game with observed actions that is continuous at infinity. If σ_i is unimprovable given σ_{-i} for all $i \in N$, then σ constitutes an SPE.

Proof allows for infinite action spaces at some stages. There exist versions for games with unobserved actions.

Proof

Suppose that σ_i is unimprovable given σ_{-i}, but σ_i is not a best response to σ_{-i} following some history h_t. Let σ_i^1 be a strictly better response and define

$$\varepsilon = u_i(\sigma_i^1, \sigma_{-i}|h_t) - u_i(\sigma_i, \sigma_{-i}|h_t) > 0.$$

Since the game is *continuous at infinity*, there exists $t' > t$ and σ_i^2 s.t. σ_i^2 is identical to σ_i^1 at all information sets up to (and including) stage t', σ_i^2 coincides with σ_i across all longer histories and

$$|u_i(\sigma_i^2, \sigma_{-i}|h_t) - u_i(\sigma_i^1, \sigma_{-i}|h_t)| < \varepsilon/2.$$

Then

$$u_i(\sigma_i^2, \sigma_{-i}|h_t) > u_i(\sigma_i, \sigma_{-i}|h_t).$$

Proof

σ_i^3: strategy obtained from σ_i^2 by replacing the stage t' actions following any history $h_{t'}$ with the corresponding actions under σ_i

Conditional on any $h_{t'}$, σ_i and σ_i^3 coincide, hence

$$u_i(\sigma_i^3, \sigma_{-i}|h_{t'}) = u_i(\sigma_i, \sigma_{-i}|h_{t'}).$$

As σ_i is *unimprovable* given σ_{-i}, and conditional on $h_{t'}$ the subsequent play in strategies σ_i and σ_i^2 differs only at stage t',

$$u_i(\sigma_i, \sigma_{-i}|h_{t'}) \geq u_i(\sigma_i^2, \sigma_{-i}|h_{t'}).$$

Then

$$u_i(\sigma_i^3, \sigma_{-i}|h_{t'}) \geq u_i(\sigma_i^2, \sigma_{-i}|h_{t'})$$

for all histories $h_{t'}$. Since σ_i^2 and σ_i^3 coincide before reaching stage t',

$$u_i(\sigma_i^3, \sigma_{-i}|h_t) \geq u_i(\sigma_i^2, \sigma_{-i}|h_t).$$

Proof

σ_i^4: strategy obtained from σ_i^3 by replacing the stage $t' - 1$ actions following any history $h_{t'-1}$ with the corresponding actions under σ_i

Similarly,

$$u_i(\sigma_i^4, \sigma_{-i}|h_t) \geq u_i(\sigma_i^3, \sigma_{-i}|h_t) \ldots$$

The final strategy $\sigma_i^{t'-t+3}$ is identical to σ_i conditional on h_t and

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}|h_t) = u_i(\sigma_i^{t'-t+3}, \sigma_{-i}|h_t) &\geq \ldots \\ &\geq u_i(\sigma_i^3, \sigma_{-i}|h_t) \geq u_i(\sigma_i^2, \sigma_{-i}|h_t) > u_i(\sigma_i, \sigma_{-i}|h_t), \end{aligned}$$

a contradiction.

Applications

Apply the single deviation principle to repeated prisoners' dilemma to implement the following equilibrium paths for high discount factors:

- $(C, C), (C, C), \ldots$
- $(C, C), (C, C), (D, D), (C, C), (C, C), (D, D), \ldots$
- $(C, D), (D, C), (C, D), (D, C) \ldots$

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

Cooperation is possible in repeated play.

Bargaining with Alternating Offers

Rubinstein (1982)

- players $i = 1, 2$; $j = 3 - i$
- set of feasible utility pairs

$$U = \{(u_1, u_2) \in [0, \infty)^2 | u_2 \leq g_2(u_1)\}$$

- g_2 s. decreasing, concave (and hence continuous), $g_2(0) > 0$
- δ_i: discount factor of player i
- at every time $t = 0, 1, \ldots$, player $i(t)$ proposes an alternative $u = (u_1, u_2) \in U$ to player $j(t) = 3 - i(t)$

$$i(t) = \begin{cases} 1 \text{ for} & t \text{ even} \\ 2 \text{ for} & t \text{ odd} \end{cases}$$

  - if $j(t)$ accepts the offer, game ends yielding payoffs $(\delta_1^t u_1, \delta_2^t u_2)$
  - otherwise, game proceeds to period $t + 1$

Stationary SPE

Define $g_1 = g_2^{-1}$. Graphs of g_2 and g_1^{-1}: Pareto-frontier of U

Let (m_1, m_2) be the unique solution to the following system of equations

$$\begin{aligned} m_1 &= \delta_1 g_1 (m_2) \\ m_2 &= \delta_2 g_2 (m_1). \end{aligned}$$

(m_1, m_2) is the intersection of the graphs of $\delta_2 g_2$ and $(\delta_1 g_1)^{-1}$.

SPE in "stationary" strategies: in any period where player i has to make an offer to j, he offers u with $u_j = m_j$ and $u_i = g_i(m_j)$, and j accepts only offers u with $u_j \geq m_j$.

Single-deviation principle: constructed strategies form an SPE.

Is the SPE unique?

Iterated Conditional Dominance

Definition 3

In a multi-stage game with observable actions, an action a_i is *conditionally dominated* at stage t given history h_t if, in the subgame starting at h_t, every strategy for player i that assigns positive probability to a_i is strictly dominated.

Proposition 1

In any multi-stage game with observable actions, every SPE survives the iterated elimination of conditionally dominated strategies.

Equilibrium uniqueness

Iterated conditional dominance: stationary equilibrium is essentially the unique SPE.

Theorem 2

The SPE of the alternating-offer bargaining game is unique, except for the decision to accept or reject Pareto-inefficient offers.

Proof

- Following a disagreement at date t, player i cannot obtain a period t expected payoff greater than

$$M_i^0 = \delta_i \max_{u \in U} u_i = \delta_i g_i(0)$$

- Rejecting an offer u with $u_i > M_i^0$ is conditionally dominated by accepting such an offer for i.
- Once we eliminate dominated actions, i accepts all offers u with $u_i > M_i^0$ from j.
- Making any offer u with $u_i > M_i^0$ is dominated for j by an offer $\bar{u} = \lambda u + (1 - \lambda)\left(M_i^0, g_j\left(M_i^0\right)\right)$ for $\lambda \in (0, 1)$ (both offers are accepted immediately).

Proof

Under the surviving strategies

- j can reject an offer from i and make a counteroffer next period that leaves him with slightly less than $g_j\left(M_i^0\right)$, which i accepts; it is conditionally dominated for j to accept any offer smaller than

$$m_j^1 = \delta_j g_j\left(M_i^0\right)$$

- i cannot expect to receive a continuation payoff greater than

$$M_i^1 = \max\left(\delta_i g_i\left(m_j^1\right), \delta_i^2 M_i^0\right) = \delta_i g_i\left(m_j^1\right)$$

  after rejecting an offer from j

$$\delta_i g_i\left(m_j^1\right) = \delta_i g_i\left(\delta_j g_j\left(M_i^0\right)\right) \geq \delta_i g_i\left(g_j\left(M_i^0\right)\right) = \delta_i M_i^0 \geq \delta_i^2 M_i^0$$

Proof

Recursively define

$$
\begin{aligned}
m_j^{k+1} &= \delta_j g_j\left(M_i^k\right) \\
M_i^{k+1} &= \delta_i g_i\left(m_j^{k+1}\right)
\end{aligned}
$$

for $i = 1, 2$ and $k \geq 1$. $(m_i^k)_{k\geq 0}$ is increasing and $(M_i^k)_{k\geq 0}$ is decreasing.

Prove by induction on k that, under any strategy that survives iterated conditional dominance, player $i = 1, 2$

- never accepts offers with $u_i < m_i^k$
- always accepts offers with $u_i > M_i^k$, but making such offers is dominated for j.

Proof

- The sequences (m_i^k) and (M_i^k) are monotonic and bounded, so they need to converge. The limits satisfy

$$\begin{aligned} m_j^\infty &= \delta_j g_j\left(\delta_i g_i\left(m_j^\infty\right)\right) \\ M_i^\infty &= \delta_i g_i\left(m_j^\infty\right). \end{aligned}$$

- (m_1^∞, m_2^∞) is the (unique) intersection point of the graphs of the functions $\delta_2 g_2$ and $(\delta_1 g_1)^{-1}$
- $M_i^\infty = \delta_i g_i\left(m_j^\infty\right) = m_i^\infty$
- All strategies of i that survive iterated conditional dominance accept u with $u_i > M_i^\infty = m_i^\infty$ and reject u with $u_i < m_i^\infty = M_i^\infty$.

Proof

In an SPE

- at any history where i is the proposer, i's payoff is at least $g_i(m_j^\infty)$: offer u arbitrarily close to $(g_i(m_j^\infty), m_j^\infty)$, which j accepts under the strategies surviving the elimination process
- i cannot get more than $g_i(m_j^\infty)$
  - any offer made by i specifying a payoff greater than $g_i(m_j^\infty)$ for himself would leave j with less than m_j^∞; such offers are rejected by j under the surviving strategies
  - under the surviving strategies, j never offers i more than $M_i^\infty = \delta_i g_i(m_j^\infty) \leq g_i(m_j^\infty)$
- hence i's payoff at any history where i is the proposer is exactly $g_i(m_j^\infty)$; possible only if i offers $(g_i(m_j^\infty), m_j^\infty)$ and j accepts with probability 1

Uniquely pinned down actions at every history, except those where j has just received an offer (u_i, m_j^∞) for some $u_i < g_i(m_j^\infty)$...

Properties of the equilibrium

- The SPE is efficient—agreement is obtained in the first period, without delay.
- SPE payoffs: $(g_1(m_2), m_2)$, where (m_1, m_2) solve

$$\begin{aligned} m_1 &= \delta_1 g_1(m_2) \\ m_2 &= \delta_2 g_2(m_1). \end{aligned}$$

- Patient players get higher payoffs: the payoff of player i is increasing in δ_i and decreasing in δ_j.
- For a fixed $\delta_1 \in (0, 1)$, the payoff of player 2 converges to 0 as $\delta_2 \to 0$ and to $\max_{u \in U} u_2$ as $\delta_2 \to 1$.
- If U is symmetric and $\delta_1 = \delta_2$, player 1 enjoys a first mover advantage: $m_1 = m_2$ and $g_1(m_2) = m_2/\delta > m_2$.

Nash Bargaining

Assume g_2 is decreasing, s. concave and continuously differentiable.

Nash (1950) bargaining solution:

$$\{u^*\} = \arg\max_{u \in U} u_1 u_2 = \arg\max_{u \in U} u_1 g_2(u_1).$$

Theorem 3 (Binmore, Rubinstein and Wolinsky 1985)

Suppose that $\delta_1 = \delta_2 =: \delta$ in the alternating bargaining model. Then the unique SPE payoffs converge to the Nash bargaining solution as $\delta \to 1$.

$$m_1 g_2(m_1) = m_2 g_1(m_2)$$

$(m_1, g_2(m_1))$ and $(g_1(m_2), m_2)$ belong to the intersection of g_2's graph with the same hyperbola, which approaches the hyperbola tangent to the boundary of U (at u^*) as $\delta \to 1$.

Bargaining with random selection of proposer

- Two players need to divide \$1.
- Every period $t = 0, 1, \ldots$ player 1 is chosen with probability p to make an offer to player 2.
- Player 2 accepts or rejects 1's proposal.
- Roles are interchanged with probability $1 - p$.
- In case of disagreement the game proceeds to the next period.
- The game ends as soon as an offer is accepted.
- Player $i = 1, 2$ has discount factor δ_i.

Equilibrium

- The unique equilibrium is stationary, i.e., each player i has the same expected payoff v_i in every subgame.
- Payoffs solve

$$\begin{aligned} v_1 &= p(1-\delta_2 v_2) + (1-p)\delta_1 v_1 \\ v_2 &= p\delta_2 v_2 + (1-p)(1-\delta_1 v_1). \end{aligned}$$

- The solution is

$$\begin{aligned} v_1 &= \frac{p/(1-\delta_1)}{p/(1-\delta_1) + (1-p)/(1-\delta_2)} \\ v_2 &= \frac{(1-p)/(1-\delta_2)}{p/(1-\delta_1) + (1-p)/(1-\delta_2)}. \end{aligned}$$

Comparative Statics

$$
\begin{aligned}
v_1 &= \frac{1}{1 + \frac{(1-p)(1-\delta_1)}{p(1-\delta_2)}} \\
v_2 &= \frac{1}{1 + \frac{p(1-\delta_2)}{(1-p)(1-\delta_1)}}.
\end{aligned}
$$

- Immediate agreement
- First mover advantage
  - v_1 increases with p, v_2 decreases with p.
  - For $\delta_1 = \delta_2$, we obtain $v_1 = p, v_2 = 1 - p$.
- Patience pays off
  - v_i increases with δ_i and decreases with δ_j ($j = 3 - i$).
  - Fix δ_j and take $\delta_i \to 1$, we get $v_i \to 1$ and $v_j \to 0$.