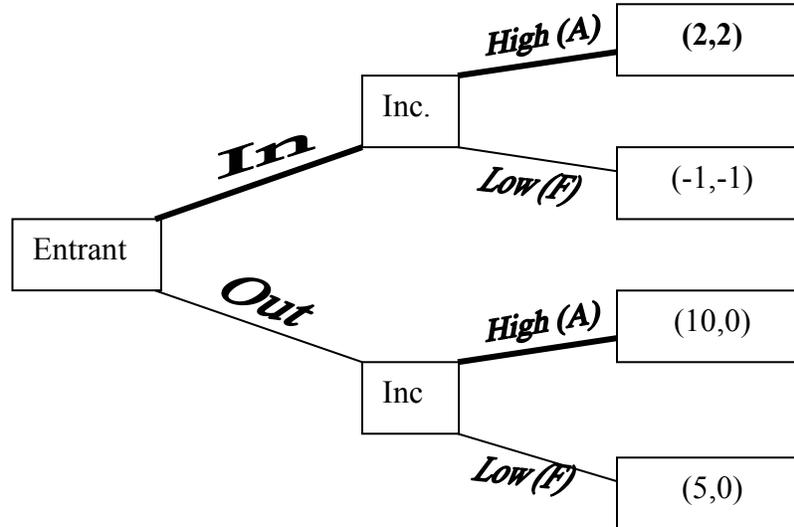


Problem Set #4 Solutions

Problem 1:

a) The extensive form of the game is as follows:



Using backwards induction, the incumbent will always set high prices, thus accommodating entry. Seeing this, the entrant will enter.

- b) No matter what happens in the first market, the second market will turn out exactly as described in part a). Rolling back to the first market, the incumbent has no ability to deter future entry, so he is forced to acquiesce and set high prices in response to entry. The first entrant, therefore, will enter. Hence, we always observe entry and high prices.
- c) Uncertainty as to the payoffs for the incumbent implies that the entrant(s) must be concerned with low prices. The threat is much more credible than before. Notationally, let $\pi^E(a)$ be the entrant's payoff when taking action a and $\pi^I(a|t)$ be the incumbent's payoff when taking action a and his type is t .
- (i) Suppose the entrant does indeed enter, then the sane incumbent will accommodate and set high prices, while the crazy incumbent will fight by setting low prices. This is a matter of profit maximization. Rolling back to the entrant's decision, if he stays out, he earns zero.

$$\pi^E(Out) = 0$$

If he enters, then with probability 0.80 he faces a sane opponent who accommodates, and with probability 0.20 he faces a crazy opponent who fights. Hence, his payoff is

$$\pi^E(In) = 0.8(2) + 0.2(-1) = \frac{7}{5}$$

The entrant will enter.

- (ii) With two markets, the sane incumbent has an incentive to pretend to be crazy by fighting in the first market; this may deter the entrant from entering. In the second market, the sane incumbent will always accommodate, and the crazy type will always

fight (see the previously given arguments). Even in the first market, the crazy type will always fight. Why? What would be the point of accommodating? The more incumbents accommodate, the more likely it is that the entrant enters, which is bad for both the sane and crazy types. Neither wants entry, but only the crazy type can credibly, in a static sense, threaten to fight. The sane type hopes that by sacrificing some short run profit in the first market by fighting he can convince the entrant that he is crazy, thus deterring entry.

There can be no pure strategy solution to this game: If the sane type always fights (a pooling equilibrium), then the second entrant gains no information by observing the incumbent fighting the first entrant. Hence, in the second market we are back to the same problem as in part (i), and the entrant enters. The incumbent thus regrets fighting in the first market; his effort to deter entry failed.

If the sane type always accommodates, then the second entrant can perfectly detect who is who. If the incumbent fights, he must be crazy; the entrant stays out. If the incumbent fights, he must be sane; the entrant enters. The sane incumbent thus wants to fight and fool the entrant.

The solution must be in mixed strategies. In any event, if the second entrant observes accommodation in the first market, then he knows that the incumbent is definitely sane. What about when fighting is observed? Let x be the probability that the sane type fights in the first market, μ be the probability that the incumbent is sane given that he fought in the first market (this belief must be consistent with Bayes' rule), and δ be the probability that the second entrant enters given that the incumbent fought in the first market:

$$\begin{aligned} x &\equiv P(F | \textit{sane}) \\ \mu &\equiv P(\textit{sane} | F) \\ \delta &\equiv P(In | F) \end{aligned}$$

First, taking x , the incumbent's strategy as given, we shall derive δ . The entrant takes δ as given, observes the incumbent fighting, and updates his/her beliefs as to the probability that the incumbent is sane. Using Bayes' rule, the updating is as follows:

$$\begin{aligned} \mu &\equiv P(\textit{sane} | F) \\ &= \frac{P(F | \textit{sane})P(\textit{sane})}{P(F | \textit{sane})P(\textit{sane}) + P(F | \textit{crazy})P(\textit{crazy})} \\ &= \frac{0.8x}{0.8x + 0.2(1)} = \frac{4x}{4x + 1} \end{aligned}$$

Consider the payoffs for the entrant upon observing fighting in the first market:

$$\begin{aligned}\pi^E(\text{Out} | F) &= 0 \\ \pi^E(\text{In} | F) &= \mu(2) + (1 - \mu)(-1) = 3\mu - 1 \\ &= \frac{12x}{4x+1} - 1 = \frac{8x-1}{4x+1}\end{aligned}$$

Hence,

x	Optimal Strategy	Induced δ
$x < 1/8$	Out	$\delta = 0$
$x = 1/8$	Indifferent	$\delta \in [0,1]$
$x > 1/8$	In	$\delta = 1$

Next, taking δ , the entrant's strategy as given, we shall derive x . Consider the payoffs for the incumbent:

$$\begin{aligned}\pi^I(A | \text{sane}) &= 2 + 2 \\ &= 4 \\ \pi^I(F | \text{sane}) &= -1 + \delta(2) + (1 - \delta)(10) \\ &= 9 - 8\delta\end{aligned}$$

Hence,

δ	Optimal Strategy	Induced x
$\delta < 5/8$	F	$x = 1$
$\delta = 5/8$	Indifferent	$x \in [0,1]$
$\delta > 5/8$	A	$x = 0$

As we suspected, no pure strategies are consistent: High δ implies low x ; but low x implies low δ . The only consistent behavior is the mixed strategy equilibrium in which the incumbent fights with probability $x = 1/8$, and the entrant enters, in response to fighting in the first market, with probability $\delta = 5/8$.

Problem 2:

- a) The pharmaceutical company solves the following problem:

$$\max_{P,A} A^{0.5}(100 - P)(P - 60) - A$$

with FOC

$$\begin{aligned}(P): A^{0.5}(100 - P)(1) + A^{0.5}(-1)(P - 60) &= 0 \\ : 100 - P &= P - 60 \\ : 2P &= 160 \\ : P &= 80\end{aligned}$$

and

$$\begin{aligned}(A): 0.5A^{-0.5}(100 - P)(P - 60) - 1 &= 0 \\ : 200A^{-0.5} &= 1 \\ : A^{0.5} &= 200 \\ : A &= 40,000\end{aligned}$$

The maximized output and profit levels are thus

$$\begin{aligned} Q &= A^{0.5}(100 - P) \\ &= 200(20) \\ &= 4,000 \end{aligned}$$

and

$$\begin{aligned} \pi &= A^{0.5}(100 - P)(P - 60) - A \\ &= 200(20)(20) - 40,000 \\ &= 80,000 - 40,000 \\ &= 40,000 \end{aligned}$$

b) The price elasticity of demand is

$$\varepsilon_P = \frac{\partial Q}{\partial P} \frac{P}{Q} = \frac{-A^{0.5}P}{Q} = \frac{-(200)(80)}{4,000} = -4$$

while the advertising elasticity of demand is

$$\varepsilon_A = \frac{\partial Q}{\partial A} \frac{A}{Q} = 0.5A^{-0.5}(100 - P) \frac{A}{A^{0.5}(100 - P)} = 0.5$$

Last, the advertising/sales ration is

$$\frac{A}{PQ} = \frac{40,000}{(80)(4,000)} = \frac{1}{8}$$

Notice that we can verify these results by using the following formula (derived in the notes):

$$\begin{aligned} \frac{-\varepsilon_A}{\varepsilon_P} &= \frac{A}{PQ} \\ &= \frac{-\left(\frac{1}{2}\right)}{(-4)} = \frac{1}{8} \end{aligned}$$

Problem 3:

a) Discussed in class.

Problem 4:

The multiperiod aspect of the problem only serves to rescale the payoffs. Given the discount factor, if single period payoffs are π , then the total payoff is 5π . Since each period is independent and identical, we can solve for the static outcome and extrapolate to the five-period solution.

a) For simplicity, I shall solve for a general solution to the Cournot problem (this should be rudimentary by now), then plug in different marginal costs to solve for the different parts. Consider the two firm asymmetric Cournot game in which firm i has marginal cost c_i ; the firm thus solves the following problem:

$$\max_{q_i} (100 - q_i - q_j - c_i)q_i$$

with FOC

$$\begin{aligned}
100 - 2q_i - q_j - c_i &= 0 \\
100 - Q - q_i - c_i &= 0 \\
\sum (100 - 2q_i - q_j - c_i) &= \sum 0 \\
200 - 2Q - Q - c_1 - c_2 &= 0 \\
Q &= \frac{200 - c_1 - c_2}{3} \\
q_i &= 100 - c_i - Q \\
&= P - c_i \\
\pi_i^{period} &= (P - c_i)q_i = q_i^2
\end{aligned}$$

Because demand is linear, we can calculate per-period consumer surplus as the area of a triangle.

$$\frac{1}{2}(P^{\max} - P)Q = \frac{1}{2}Q^2$$

where P^{\max} is the price-intercept of the inverse demand function.

- (i) When both firms are operating with marginal and average costs of \$15, the Cournot solution is

$$Q = \frac{200 - 15 - 15}{3} = \frac{170}{3}$$

$$P = 100 - \frac{170}{3} = \frac{130}{3}$$

$$q_1 = q_2 = \frac{85}{3}$$

$$\pi^{period} = \left(\frac{85}{3}\right)^2$$

$$\pi^{total} = 5\left(\frac{85}{3}\right)^2$$

Additionally, this leads to consumer surplus of

$$CS = 5 * \frac{1}{2} \left(\frac{170}{3}\right)^2 = 8027.77$$

- (ii) Suppose Firm 1 develops the new technology, what is the new outcome?

$$Q = \frac{200 - 10 - 15}{3} = \frac{175}{3}$$

$$P = 100 - \frac{175}{3} = \frac{125}{3}$$

$$q_1 = \frac{125}{3} - 10 = \frac{95}{3}$$

$$q_2 = \frac{125}{3} - 15 = \frac{80}{3}$$

$$\pi_1^{period} = \left(\frac{95}{3}\right)^2$$

$$\pi_1^{total} = 5\left(\frac{95}{3}\right)^2$$

$$\pi_2^{period} = \left(\frac{80}{3}\right)^2$$

$$\pi_2^{total} = 5\left(\frac{80}{3}\right)^2$$

If the firm does not develop the new technology, Firm 1's profits are

$$5\left(\frac{85}{3}\right)^2$$

but, by developing it, he makes profits of

$$5\left(\frac{95}{3}\right)^2 - Cost$$

The firm is willing to develop the technology as long as

$$5\left(\frac{95}{3}\right)^2 - Cost \geq 5\left(\frac{85}{3}\right)^2$$

$$Cost \leq 5\left(\frac{95}{3}\right)^2 - 5\left(\frac{85}{3}\right)^2$$

Hence, the maximum willingness to pay is

$$5\left(\frac{95}{3}\right)^2 - 5\left(\frac{85}{3}\right)^2 = 1000$$

Consumer surplus is

$$CS = 5 * \frac{1}{2} \left(\frac{175}{3}\right)^2 \approx 8506.94$$

- (iii) The social value of the new technology is merely the change in consumer surplus plus the change in aggregate profits.

$$\begin{aligned}
 SValue &= \Delta CS + \Delta \Sigma \pi \\
 &= 5 \left\{ \left[\frac{1}{2} \left(\frac{175}{3} \right)^2 - \frac{1}{2} \left(\frac{170}{3} \right)^2 \right] + \left[\left(\frac{95}{3} \right)^2 - \left(\frac{85}{3} \right)^2 \right] + \left[\left(\frac{80}{3} \right)^2 - \left(\frac{85}{3} \right)^2 \right] \right\} \\
 &= \frac{18375}{18} \approx 1020.83
 \end{aligned}$$

(iv) If Firm 2 develops the copycat technology, then the Cournot outcome is

$$Q = \frac{200 - 10 - 10}{3} = 60$$

$$P = 100 - 60 = 40$$

$$q_1 = q_2 = 30$$

$$\pi^{period} = 30^2 = 900$$

$$\pi^{total} = 4,500$$

By the same reasoning as in part (ii), Firm 2's maximum willingness to pay is

$$4,500 - 5 \left(\frac{80}{3} \right)^2 = \frac{8500}{9} \approx 944.44$$

Also, consumer surplus is

$$CS = 5 * \frac{1}{2} (60)^2 = 9000$$

(v) The social value of the copycat technology is

$$\begin{aligned}
 SValue &= 5 \left\{ \left[1800 - \frac{1}{2} \left(\frac{175}{3} \right)^2 \right] + \left[900 - \left(\frac{95}{3} \right)^2 \right] + \left[900 - \left(\frac{80}{3} \right)^2 \right] \right\} \\
 &= \frac{16625}{18} \approx 923.61
 \end{aligned}$$

b) For the undifferentiated Bertrand game, in equilibrium firms bid the price down to the second highest marginal price.

(i) When both firms are operating with marginal and average costs of \$15, the Bertrand outcome is

$$P = 15$$

$$Q = 85$$

$$q_1 = q_2 = \frac{85}{2}$$

$$\pi^{period} = 0$$

$$\pi^{total} = 0$$

The consumer surplus is

$$CS = 5 * \frac{1}{2} (100 - 15) 85 = \frac{36125}{2} = 18062.5$$

- (ii) If Firm 1 develops the new technology, it has lower marginal costs and can undercut Firm 2's price of 15 by an arbitrarily small amount and steal the whole market. Hence, the outcome is

$$\begin{aligned}
 P &= 15 \\
 Q &= 85 \\
 q_1 &= 85 \\
 q_2 &= 0 \\
 \pi_1^{period} &= (15 - 10)85 = 425 \\
 \pi_1^{total} &= 2125 \\
 \pi_2^{period} &= 0 \\
 \pi_2^{total} &= 0
 \end{aligned}$$

Firm 1's maximum willingness to pay, or his incentive to develop is 2125. The consumer surplus is unchanged.

- (iii) The social value of developing the new technology is 2125. Consumer surplus is unchanged, as is the profit to Firm 2. The only value comes through the profits to Firm 1.
- (iv) If Firm 2 develops a copycat technology, then firms bid the price down to 10, and the outcome is

$$\begin{aligned}
 P &= 10 \\
 Q &= 90 \\
 q_1 &= q_2 = 45 \\
 \pi^{period} &= 0 \\
 \pi^{total} &= 0
 \end{aligned}$$

and consumer surplus is

$$CS = 5 * \frac{1}{2} (100 - 10)90 = 20250$$

- (v) The social value of developing the copycat technology is
 $20250 - 18062.5 - 2125 = 62.5$
 Consumers gain; Firm 1 loses; and Firm 2 is unchanged.

- c) Incentives to develop the new technology can be summarized in the following chart:

	Competitive Value	Social Value
Cournot	1000	1020.85
Bertrand	2125	2125

Bertrand competition provides higher private and social incentives to innovate. The increased private incentive comes from the fact that lower marginal benefits the innovator one-for-one. A dollar in marginal cost savings is a dollar of profit; additionally, the innovator can steal the whole market. In Cournot competition, on the other hand, a dollar in marginal cost savings is not a dollar of profit due to the price response. The increased social benefit follows from the same argument.

d) Incentives to develop the copycat technology can be summarized in the following chart:

	Competitive Value	Social Value
Cournot	944.44	923.61
Bertrand	0	62.5

Cournot competition, however, provides stronger incentives, both privately and socially, to develop copycat technology. Bertrand competition provides no private incentive, zero profits versus zero profits.

Problem 5:

a) Each firm solves the following problem:

$$\max_{q_i} (20 - q_i - q_j - 2)q_i$$

with FOC

$$18 - 2q_i - q_j = 0$$

Hence, the firm's reaction function is

$$q_i(q_j) = \frac{1}{2}(18 - q_j)$$

Using symmetry,

$$3q_i = 18$$

$$q_i = 6$$

$$Q = 6 + 6 = 12$$

$$P = 20 - 12 = 8$$

$$\pi = (8 - 2)6 = 36$$

b) Paying a manager a share of profits will not change the equilibrium. A manager will maximize his own payoff $\alpha\pi$. This will have the following FOC:

$$\alpha \frac{\partial \pi}{\partial q_i} = 0$$

which reduces to

$$\frac{\partial \pi}{\partial q_i} = 0$$

which is the original FOC.

c) The manager for Firm 2 now solves the following problem:

$$\max_{q_2} \beta(20 - q_1 - q_2)q_2$$

with FOC

$$\beta(20 - q_1 - 2q_2) = 0$$

Hence, Firm 2's reaction function is

$$q_2(q_1) = \frac{1}{2}(20 - q_1)$$

Firm 1's reaction function is unchanged. Solving this system of equations, we get the following outcome:

$$Q = \frac{38}{3}$$

$$P = \frac{22}{3}$$

$$q_1 = \frac{16}{3}$$

$$q_2 = \frac{22}{3}$$

$$\pi_1 = \left(\frac{16}{3}\right)^2 \approx 28.44$$

$$\pi_2 = \frac{352}{9} \approx 39.11$$

By paying the manager according to revenue, the firm effectively commits to over-produce; the manager behaves as though marginal cost were zero, not two. Hence, like a Stackelberg-leader, the firm does better by committing. Firm 1's shareholders will want to follow suit and switch to the new compensation scheme as well; commit to over-produce in order to boost profits. If you run the numbers, Firm 1 can boost its profits to 31.11.

- d) If the firms compete in a differentiated products Bertrand game, the firm switching to the revenue-based compensation scheme will over-produce, since the manager will still behave as though marginal cost is zero. As you may recall, leaders in pricing games tend to do worse, while followers do better; this is in direct contrast to quantity games. Because the compensation scheme induces the manager to behave as though marginal cost has been reduced, we can interpret this as an over-investment in cost-reducing technology. Cost reducing technology makes a firm tough. When firms compete in quantities, strategic substitutes, over-investment is optimal (Top Dog), as we saw in part c). On the other hand, when firms compete in prices, strategic complements, under investment is optimal (Puppy Dog).