# 14.27 Problem Set 2

Due 10/1

## Question 1: Biased Beliefs

Recall the model of auctions vs. posted prices from class. We have $N$ consumers who each have a valuation that is independent and identically distributed from a Uniform distribution. The seller has 1 product to sell. We will compare the effects of using either a posted price mechanism, and a second price auction in the case where the seller has biased beliefs.

**a)** Assume that buyers valuations are iid $F = U[0, 0.5]$, but the seller falsely believes that they are iid $\sim F\text{'} = U[0, 1]$.

*In general, when setting a posted price the firm solves:*

$$\max_p p \cdot \Pr\left[p < v_{(N)}\right] = \max_p p \cdot \left(1 - F(p)^N\right)$$

*where* $F(\cdot)$ *is the CDF of the relevant distribution of valuations. The expected value of an auction is:*

$$E(\pi(N)) = (N-1)N \int F(x)^{N-2}(1 - F(x))\, x dx$$

Calculate the following:

- $P*(F\text{'})$: The optimal price the firm will charge under biased beliefs.
  *In this case,* $F(x) = x$*. Then the problem is:* $\max_p p\left(1 - p^N\right) \Rightarrow 1 - (N+1)p^N = 0$
  $\Rightarrow P*(F\text{'}) = \left(\frac{1}{N+1}\right)^{\frac{1}{N}}$

- $P*(F)$: The optimal price to charge under correct beliefs.
  *In this case,* $F(x) = 2x$*. Then the problem is:* $\max_p p\left(1 - (2p)^N\right) \Rightarrow 1 - 2^N(N+1)p^N = 0$
  $\Rightarrow P*(F) = \frac{1}{2}\left(\frac{1}{(N+1)}\right)^{\frac{1}{N}}$

- $E_{F\text{'}}[\pi_{Posted}(F\text{'})]$: The expected profits of the posted price under biased beliefs from the point of view of the firm.
  $E_{F\text{'}}[\pi_{Posted}(F\text{'})] = P*(F\text{'})\left(1 - F'(P*(F\text{'}))^N\right) = \left(\frac{1}{N+1}\right)^{\frac{1}{N}}\left(1 - \left(\frac{1}{N+1}\right)\right) = \left(\frac{1}{N+1}\right)^{\frac{1}{N}} \frac{N}{N+1}$

| | Posted Price | | | | | Auction | |
|---|---|---|---|---|---|---|---|
| N | Biased Price | Optimal Price | Expected Profits, Biased Price | True Profits, Biased Price | True Profits, Optimal Price | Expected Profits, Biased Beliefs | Expected Profits, Correct Beliefs |
| 2 | 0.69 | 0.35 | 0.46 | 0 | 0.17 | 0.33 | 0.17 |
| 3 | 0.71 | 0.35 | 0.53 | 0 | 0.18 | 0.50 | 0.25 |
| 4 | 0.72 | 0.36 | 0.58 | 0 | 0.18 | 0.60 | 0.30 |
| 5 | 0.74 | 0.37 | 0.62 | 0 | 0.19 | 0.67 | 0.33 |
| 6 | 0.76 | 0.38 | 0.65 | 0 | 0.19 | 0.71 | 0.36 |
| 7 | 0.77 | 0.39 | 0.67 | 0 | 0.19 | 0.75 | 0.38 |
| 8 | 0.78 | 0.39 | 0.70 | 0 | 0.20 | 0.78 | 0.39 |
| 9 | 0.79 | 0.40 | 0.71 | 0 | 0.20 | 0.80 | 0.40 |
| 10 | 0.80 | 0.40 | 0.73 | 0 | 0.20 | 0.82 | 0.41 |
| 11 | 0.81 | 0.41 | 0.75 | 0 | 0.20 | 0.83 | 0.42 |
| 12 | 0.82 | 0.41 | 0.76 | 0 | 0.21 | 0.85 | 0.42 |
| 13 | 0.83 | 0.41 | 0.77 | 0 | 0.21 | 0.86 | 0.43 |
| 14 | 0.83 | 0.42 | 0.78 | 0 | 0.21 | 0.87 | 0.43 |
| 15 | 0.84 | 0.42 | 0.79 | 0 | 0.21 | 0.88 | 0.44 |
| 16 | 0.85 | 0.42 | 0.80 | 0 | 0.21 | 0.88 | 0.44 |
| 17 | 0.85 | 0.43 | 0.80 | 0 | 0.21 | 0.89 | 0.44 |
| 18 | 0.86 | 0.43 | 0.81 | 0 | 0.21 | 0.89 | 0.45 |
| 19 | 0.86 | 0.43 | 0.82 | 0 | 0.22 | 0.90 | 0.45 |
| 20 | 0.87 | 0.43 | 0.82 | 0 | 0.22 | 0.90 | 0.45 |

Figure 1: Profit Comparison

- $E_F[\pi_{Posted}(F')]$: The expected profits of the posted price under biased beliefs but from the point of view of what will actually happen.
  $E_F[\pi_{Posted}(F')] = P*(F')\left(1-F\left(P*(F')\right)^N\right) = P*(F')\max\left(1-\left(2P*(F')\right)^N, 0\right)$
  $= \left(\frac{1}{(N+1)}\right)^{\frac{1}{N}} \max\left(1-\frac{2^N}{(N+1)}, 0\right)$
- $E_{F'}[\pi_{Auction}]$: The expected profits from the auction from the point of view of the firm.
  $E_{F'}[\pi_{Auction}] = (N-1)\,N\int x^{N-2}(1-x)\,xdx = (N-1)\,N\int x^{N-1}-x^N dx = \frac{N-1}{N+1}$
- $E_F[\pi_{Auction}]$ The actual expected profits from the auction.
  *In this case, $f(x)=2$ and $F(x)=2x$. Then:*
  $E_{F'}[\pi_{Auction}] = (N-1)\,N\int 2\,(2x)^{N-2}(1-2x)\,xdx = 2^{N-1}(N-1)\,N\int_0^{0.5}\left(x^{N-1}-2x^N\right)dx$
  $= 2^{N-1}(N-1)\,N\left[\frac{x^N}{N}-2\frac{x^{N+1}}{N+1}\right]_0^{0.5} = \frac{1}{2}\frac{N-1}{N+1}$

Analyze the relation between all of these expected profits as a function of $N$, the number of bidders. It may be useful to create a table as we did in class plotting the value for different values of $N$. Identify when the firm chooses to use a posted price vs. an auction: does that always work out well for the firm?

*Note that the true profits under the biased optimal price are always 0! This is in contrast to the* *auction, where under true and biased beliefs profits are always positive. The firm chooses an auction for $N \geq 4$, and really should pick an auction for $N \geq 3$.*

**b)** Repeat the exercise above using $F\text{'} = U[0,2]$.

- $P*(F\text{'})$: The optimal price the firm will charge under biased beliefs.
  *In this case,* $F(x) = \frac{1}{2}x$. *Then the problem is:* $\max_p p\left(1 - \left(\frac{p}{2}\right)^N\right) \Rightarrow 1 - \frac{(N+1)}{2^N}p^N = 0$
  $\Rightarrow P*(F\text{'}) = 2\left(\frac{1}{N+1}\right)^{\frac{1}{N}}$

- $E_{F\text{'}}[\pi_{Posted}(F\text{'})]$: The expected profits of the posted price under biased beliefs from the point of view of the firm.
  $E_{F\text{'}}[\pi_{Posted}(F\text{'})] = P*(F\text{'})\left(1 - F'(P*(F\text{'}))^N\right) = 2\left(\frac{1}{N+1}\right)^{\frac{1}{N}}\left(\frac{N}{N+1}\right)$

- $E_F[\pi_{Posted}(F\text{'})]$: The expected profits of the posted price under biased beliefs but from the point of view of what will actually happen.
  $E_F[\pi_{Posted}(F\text{'})] = P*(F\text{'})\left(1 - F(P*(F\text{'}))^N\right) = 2\left(\frac{1}{(N+1)}\right)^{\frac{1}{N}} \max\left(1 - 4^N\left(\frac{1}{N+1}\right), 0\right)$

- $E_{F'}[\pi_{Auction}]$: The expected profits from the auction from the point of view of the firm.
  $E_{F'}[\pi_{Auction}] = (N-1)N\int_0^2 \frac{x}{2}^{N-2}\left(1 - \frac{x}{2}\right)x\frac{1}{2}dx = (N-1)N\left(\frac{1}{2}\right)^{N-1}\int_0^2 x^{N-1} - \frac{x^N}{2}dx$
  $= 2\frac{N-1}{N+1}$

- $E_F[\pi_{Auction}]$ The actual expected profits from the auction.
  *These do not change:* $\frac{1}{2}\frac{N-1}{N+1}$

| | Posted Price | | | | | Auction | |
|---|---|---|---|---|---|---|---|
| N | Biased Price | Optimal Price | Expected Profits, Biased Price | True Profits, Biased Price | True Profits, Optimal Price | Expected Profits, Biased Beliefs | Expected Profits, Correct Beliefs |
| 2 | 1.39 | 0.69 | 0.92 | 0 | 0.35 | 0.67 | 0.33 |
| 3 | 1.41 | 0.71 | 1.06 | 0 | 0.35 | 1.00 | 0.50 |
| 4 | 1.45 | 0.72 | 1.16 | 0 | 0.36 | 1.20 | 0.60 |
| 5 | 1.48 | 0.74 | 1.24 | 0 | 0.37 | 1.33 | 0.67 |
| 6 | 1.51 | 0.76 | 1.30 | 0 | 0.38 | 1.43 | 0.71 |
| 7 | 1.54 | 0.77 | 1.35 | 0 | 0.39 | 1.50 | 0.75 |
| 8 | 1.57 | 0.78 | 1.39 | 0 | 0.39 | 1.56 | 0.78 |
| 9 | 1.59 | 0.79 | 1.43 | 0 | 0.40 | 1.60 | 0.80 |
| 10 | 1.61 | 0.80 | 1.46 | 0 | 0.40 | 1.64 | 0.82 |
| 11 | 1.63 | 0.81 | 1.49 | 0 | 0.41 | 1.67 | 0.83 |
| 12 | 1.64 | 0.82 | 1.52 | 0 | 0.41 | 1.69 | 0.85 |
| 13 | 1.66 | 0.83 | 1.54 | 0 | 0.41 | 1.71 | 0.86 |
| 14 | 1.67 | 0.83 | 1.56 | 0 | 0.42 | 1.73 | 0.87 |
| 15 | 1.68 | 0.84 | 1.58 | 0 | 0.42 | 1.75 | 0.88 |
| 16 | 1.69 | 0.85 | 1.59 | 0 | 0.42 | 1.76 | 0.88 |
| 17 | 1.70 | 0.85 | 1.61 | 0 | 0.43 | 1.78 | 0.89 |
| 18 | 1.71 | 0.86 | 1.62 | 0 | 0.43 | 1.79 | 0.89 |
| 19 | 1.72 | 0.86 | 1.64 | 0 | 0.43 | 1.80 | 0.90 |
| 20 | 1.73 | 0.87 | 1.65 | 0 | 0.43 | 1.81 | 0.90 |

Figure 2: Profit Comparison