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14.30 Introduction to Statistical Methods in Economics
Spring 2009

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14.30 Exam 2

Spring 2008

Instructions: This exam is closed-book and closed-notes. You may use a calculator. Please read through the exam first in order to ask clarifying questions and to allocate your time appropriately. In order to receive partial credit in the case of computational errors, please show all work. You have approximately 85 minutes to complete the exam. Good luck!

1. (24 points) Short Questions

Don't spend much time on these questions, a short answer to each suffices.

- (a) Given a random variable X , define the standardization Z of X and derive its variance.
- (b) Suppose \bar{X}_{25} is the mean of a sample of $n = 25$ i.i.d. observations with $X_i \sim N(1, 4)$. What is the probability that the sample mean \bar{X}_{25} is between 0.5 and 1.5?

True/false/uncertain: Give a brief explanation (preferably using some algebra) if true, or try to correct the statement if false.

- (c) Flights operated by Phoenix-based Sunshine Airlines have longer average delays at every major airport in the US than Seattle-based Air Grey Skies. Therefore, the expected delay of a randomly chosen flight by Sunshine Airlines is longer than that of a randomly chosen Air Grey Skies flight.
- (d) The Central Limit Theorem says that in a large sample of i.i.d. observations X_1, \dots, X_n , the X_i 's approximately follow a normal distribution.

2. (20 points)

Suppose X and Y are random variables which are not necessarily independent, and $0 \leq p \leq 1$

- (a) Using *only the definitions* of variances and covariances, show that

$$\text{Var}(pX + (1-p)Y) = p^2\text{Var}(X) + 2p(1-p)\text{Cov}(X, Y) + (1-p)^2\text{Var}(Y)$$

(for this problem, you are not supposed to use properties of variances derived in class).

- (b) What is the variance of $pX + (1-p)Y$ in terms of the variances of X and Y and the correlation coefficient $\rho(X, Y)$?
- (c) Suppose $\text{Var}(X) = \text{Var}(Y)$. Is the variance of $pX + (1-p)Y$ greater or less than that of X ?
- (d) Now suppose that X and Y are scores from two different Math tests. You are interested in some underlying "math ability", and the two scores are noisy (and possibly correlated) measurements with $\mathbb{E}[X] = \mathbb{E}[Y] = \mu$, $\text{Var}(X) = \sigma_X^2$, $\text{Var}(Y) = \sigma_Y^2$, and $\text{Cov}(X, Y) = \sigma_{XY}$. Instead of using only one of the measurements, you decide to combine them into a weighted average $pX + (1-p)Y$ instead. What is the expectation of this weighted average? Which value of p minimizes the variance of the weighted average?

3. (10 points)

- (a) According to Chebyshev's Inequality, how does the probability of the event $(|X - \mathbb{E}[X]| > \varepsilon)$ relate to the variance $\text{Var}(X)$ of the random variable X ?
- (b) You and your friend are planning to meet on Inman Square at 7:30pm. Both of you try to be on time, and the times at which each of you arrives are independent of another. Also assume that the standard deviation σ of the arrival time (measured in minutes) is the same for both of you. What can σ at most be so that with probability of at least 92 percent, none of you has to wait for the other for more than 10 minutes?

4. (21 points)

You are working with a supercomputer back in the 1960s which operates using punch cards. For this type of machines, a piece of code is punched into the cards, where a bit is read as "1" if a hole is stamped out in the corresponding position on the card, and otherwise it's read as "0". Before you feed the code into the computer, another machine has to copy it on a different format of punch cards. This process does not work entirely error-free, and for a code of length n bits, the number X of copying errors is distributed as a Poisson random variable with arrival rate $\lambda = \frac{n}{50000}$, i.e. $X \sim P\left(\frac{n}{50000}\right)$.

- (a) One way of thinking about the problem is as follows: when copying a code of n bits, for every bit, there is a constant, but small, probability p that the copying machine flips a zero for a one and vice versa. If the copying errors are independent across different bits, what is the distribution of the number of errors? If n is very large, what justifies the use of the Poisson distribution suggested above instead? - please be as precise as you can.

Now suppose the only copying error the machine can make is not to punch out the positions corresponding to "1"s properly, e.g. because a piece of cardboard remains stuck in the hole, so that the computer will read it as a "0" instead of a "1". On the other hand all "0"s will be copied correctly. We'll assume that in a code of length n , the number of "1"s Y follows a binomial distribution with the probability of a "1" being $p = \frac{1}{2}$.

- (b) Derive the expectation and the variance of the number X of copying errors for a code of length n . Show your work. Hint: Remember that for a Poisson random variable X with arrival rate λ , the p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbb{E}[X] = \text{Var}(X) = \lambda$.

5. (15 points)

Suppose, $X \sim U[0, 1]$ is uniformly distributed on the interval $[0, 1]$, and Y has p.d.f.

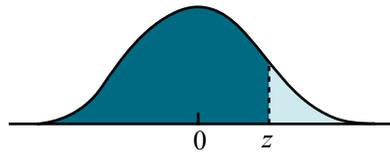
$$f_Y(y) = \begin{cases} \lambda e^{\lambda y} & \text{if } y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Give a function $u(\cdot)$ such that $Y = u(X)$ (hint: what is the c.d.f. of Y ?).

6. (10 points)

Suppose $\text{Var}(X) = \text{Var}(Y) = 1$ and $\mathbb{E}[X] = 2$ and $\mathbb{E}[Y] = 1$, and the correlation coefficient is $\rho(X, Y) = \frac{1}{3}$. What is the expectation of the squared difference between the two random variables, $\mathbb{E}[(X - Y)^2]$?

Cumulative areas under the standard normal distribution



(Cont.)

z	0	1	2	3	4	5	6	7	8	9
-3	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0126	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0238	0.0233
-1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0300	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0570	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3112
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

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Cumulative areas under the standard normal distribution

(Cont.)

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9648	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Source: B. W. Lindgren, *Statistical Theory* (New York: Macmillan, 1962), pp. 392-393.

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