

Lecture 12

Confidence Sets

1 Introduction

So far, we have been considering point estimation. In this lecture, we will study interval estimation. Let X denote our data. Let $\theta \in \mathbb{R}$ be our parameter of interest. Our task is to construct a data-dependent interval $[l(X), r(X)]$ so that it contains θ with large probability. One possibility is to set $l(X) = -\infty$ and $r(X) = +\infty$. Such an interval will contain θ with probability 1. Of course, the problem with this interval is that it is too long. So, we want to construct an interval as short as possible.

More generally, instead of intervals, we can consider confidence set $C(X) \subset \mathbb{R}$ such that it contains θ with large probability and has as small volume as possible. The concept of confidence sets can be also applied for any set of possible parameter values Θ , not just for \mathbb{R} .

Let us introduce basic concepts related to confidence sets.

Definition 1. Coverage probability of the set $C(X) \subset \Theta$ is the probability (under the assumption that the true value is θ) that confidence set $C(X)$ contains θ , i.e. Coverage Probability(θ) = $P_\theta\{\theta \in C(X)\}$.

Of course, in practice, we are interested in confidence sets that contain the true parameter value with large probability uniformly over the set of possible parameters values.

Definition 2. Confidence level is the minimum of coverage probabilities over the set of possible parameter values, i.e. Confidence Level = $\inf_{\theta \in \Theta} P_\theta\{\theta \in C(X)\}$. We say that the confidence set $C(X)$ has confidence level α if $\inf_{\theta \in \Theta} P_\theta\{\theta \in C(X)\} \geq \alpha$.

Let us consider how we can construct confidence sets.

2 Test Inversion

For each possible parameter value $\theta_0 \in \Theta$, consider the problem of testing the null hypothesis, $H_0 : \theta = \theta_0$ against the alternative, $H_a : \theta \neq \theta_0$. Suppose that for each such hypothesis we have a test of size α . Then the confidence set $C(X) = \{\theta_0 \in \Theta : \text{the null hypothesis that } \theta = \theta_0 \text{ is not rejected}\}$ is of the confidence level $1 - \alpha$. Indeed, suppose that the true value of parameter is θ_0 . Since the test of $\theta = \theta_0$ against $\theta \neq \theta_0$ has level α by construction, $P_{\theta_0}\{\text{the test rejects } \theta = \theta_0\} \leq \alpha$. So, with probability at least $1 - \alpha$, $\theta_0 \in C(X)$. In other words, $P_{\theta_0}\{\theta_0 \in C(X)\} \geq 1 - \alpha$. The same holds for all $\theta_0 \in \Theta$. So, $\inf_{\theta \in \Theta} P_\theta\{\theta \in C(X)\} \geq 1 - \alpha$.

This procedure is called test inversion. One problem with the test inversion is that sometimes a confidence set obtained via this procedure will consist of several disjoint intervals which is unattractive from an applied perspective.

Note that if we have a way to construct a confidence set, we can use it for testing. Indeed, once we have a confidence set $C(X)$ of level $1 - \alpha$, we can form a test of the null hypothesis, H_0 , that $\theta = \theta_0$ against the alternative, H_a , that $\theta \neq \theta_0$ by accepting the null hypothesis if and only if $\theta_0 \in C(X)$. This test will have size α .

Example 1 Let X_1, \dots, X_n be a random sample from distribution $N(\mu, 1)$. Let us use the test inversion to construct a confidence set for μ of level $1 - \alpha$. Let us consider the problem of testing the null hypothesis, $H_0 : \mu = \mu_0$ against the alternative, $H_a : \mu \neq \mu_0$. Under the null hypothesis, $\sqrt{n}(\bar{X}_n - \mu_0) \sim N(0, 1)$. One possible test of size α is to accept the null hypothesis if and only if $z_{\alpha/2} \leq \sqrt{n}(\bar{X}_n - \mu_0) \leq z_{1-\alpha/2}$. This test will accept the null hypothesis $\mu = \mu_0$ if and only if $\bar{X}_n - z_{1-\alpha/2}/\sqrt{n} \leq \mu_0 \leq \bar{X}_n - z_{\alpha/2}/\sqrt{n}$. So, the confidence set is $[\bar{X}_n - z_{1-\alpha/2}/\sqrt{n}, \bar{X}_n - z_{\alpha/2}/\sqrt{n}]$. Note that we actually get an interval in this example.

Example 2 Let X_1, \dots, X_n be a random sample from the distribution $N(\mu, \sigma^2)$. Let us use the test inversion to construct a confidence set for σ^2 of level $1 - \alpha$. Consider the problem of testing the null hypothesis, $H_0 : \sigma^2 = \sigma_0^2$ against the alternative, $H_a : \sigma^2 \neq \sigma_0^2$. Under the null hypothesis, $(n - 1)s^2/\sigma_0^2 \sim \chi^2(n - 1)$. One possible test of size α is to accept the null hypothesis if and only if

$$\chi_{\alpha/2}^2(n - 1) \leq (n - 1)s^2/\sigma_0^2 \leq \chi_{1-\alpha/2}^2(n - 1)$$

This test will accept the null hypothesis $\sigma^2 = \sigma_0^2$ if and only if

$$(n - 1)s^2/\chi_{1-\alpha/2}^2(n - 1) \leq \sigma_0^2 \leq (n - 1)s^2/\chi_{\alpha/2}^2(n - 1)$$

So, the confidence set is

$$\left[\frac{(n - 1)s^2}{\chi_{1-\alpha/2}^2(n - 1)}, \frac{(n - 1)s^2}{\chi_{\alpha/2}^2(n - 1)} \right]$$

In general, if we can find a pivotal quantity $Q = q(X_1, \dots, X_n, \theta_0)$ such that the distribution of Q under the null hypothesis $\theta = \theta_0$ does not depend on the choice of θ_0 , then we can use Q for testing and confidence set construction. Indeed, since the distribution of Q is independent of the true parameter value, we can find numbers a and b such that $P_{\theta_0}\{a \leq Q \leq b\} = 1 - \alpha$ for all $\theta_0 \in \Theta$. Then one possible test is to accept the null hypothesis that $\theta = \theta_0$ if and only if $a \leq q(X_1, \dots, X_n, \theta_0) \leq b$. The confidence set will consist of all parameter values θ_0 which are accepted.

3 Pratt's Theorem

Informally, the theorem states that if we use a uniformly most powerful test (UMP) for the confidence set construction, the expected length of the confidence set will be the shortest among all confidence sets of a

given level.

Theorem 3. *Let $X \sim f(x|\theta)$ be our data. Let $C(X)$ be our confidence set for θ . Then, under some regularity conditions, for any θ_0 ,*

$$E_{\theta_0}[\text{length of } C(X)] = \int P_{\theta_0}\{\theta \in C(X)\}d\theta$$

Moreover,, if $C(X)$ is constructed by inverting a UMP test of size α , then $C(X)$ has the shortest expected length among all confidence sets of level $1 - \alpha$ for any θ_0 .

Proof. The first result follows from

$$\begin{aligned} E_{\theta_0}[\text{length of } C(X)] &= E_{\theta_0}\left[\int_{\theta} I\{\theta \in C(X)\}d\theta\right] \\ &= \int_x \int_{\theta} I\{\theta \in C(X)\}d\theta f(x|\theta_0)dx \\ &= \int_{\theta} \int_x I\{\theta \in C(X)\}f(x|\theta_0)dx d\theta \\ &= \int P_{\theta_0}\{\theta \in C(X)\}d\theta \end{aligned}$$

Note that $\int P_{\theta_0}\{\theta \in C(X)\}d\theta = \int_{\theta \neq \theta_0} P_{\theta_0}\{\theta \in C(X)\}d\theta$ and, for any $\theta \neq \theta_0$, $P_{\theta_0}\{\theta \in C(X)\}$ equals 1 minus power of the test based on confidence set $C(X)$. So, if $\tilde{C}(X)$ denotes the confidence set constructed by inverting a UMP test,

$$P_{\theta_0}\{\theta \in C(X)\} \geq P_{\theta_0}\{\theta \in \tilde{C}(X)\}$$

and

$$\int P_{\theta_0}\{\theta \in C(X)\}d\theta \geq \int P_{\theta_0}\{\theta \in \tilde{C}(X)\}d\theta$$

Combining this inequality with the first result yields the second result of the theorem. \square

Example 3 Let X_1, \dots, X_n be a random sample from distribution $N(\mu, \sigma^2)$. We have already seen that the UMP test of the null hypothesis, H_0 , that $\mu = \mu_0$ against the alternative, H_a , that $\mu \neq \mu_0$ accepts the null hypothesis if and only if $|(\bar{X}_n - \mu)/\sqrt{s^2/n}| \leq t_{1-\alpha/2}(n-1)$. So, the confidence interval with shortest expected length is

$$\left[\bar{X}_n - \frac{s}{\sqrt{n}}t_{1-\alpha/2}, \bar{X}_n + \frac{s}{\sqrt{n}}t_{1-\alpha/2} \right]$$

4 Asymptotic Theory for Interval Construction

Let X_1, \dots, X_n be a random sample from distribution $f(x|\theta)$ with $\theta \in \Theta$. Under some regularity conditions,

$$\sqrt{n}(\hat{\theta}_{ML} - \theta) \Rightarrow N(0, I^{-1}(\theta))$$

For any function $h : \Theta \rightarrow \mathbb{R}$, under some regularity conditions, by delta-method,

$$\sqrt{n}(h(\hat{\theta}_{ML}) - h(\theta)) \Rightarrow N(0, (h'(\theta))^2 I^{-1}(\theta))$$

We can consistently estimate $(h'(\theta))^2 I^{-1}(\theta)$ by $n(h'(\hat{\theta}_{ML}))^2(-\partial^2 l_n(\hat{\theta}_{ML})/\partial\theta^2)^{-1}$. Denote

$$\hat{V}(h(\hat{\theta}_{ML})) = (h'(\hat{\theta}_{ML}))^2(-\partial^2 l_n(\hat{\theta}_{ML})/\partial\theta^2)^{-1}$$

By the Slutsky theorem,

$$\frac{h(\hat{\theta}_{ML}) - h(\theta)}{\sqrt{\hat{V}(h(\hat{\theta}_{ML}))}} \Rightarrow N(0, 1)$$

So, we can construct a confidence interval for $h(\theta)$ as

$$\left[h(\hat{\theta}_{ML}) + z_{\alpha/2} \sqrt{\hat{V}(h(\hat{\theta}_{ML}))}, h(\hat{\theta}_{ML}) + z_{1-\alpha/2} \sqrt{\hat{V}(h(\hat{\theta}_{ML}))} \right]$$

Note that this confidence set is essentially constructed based on the Wald statistic.

Example 4 Let X_1, \dots, X_n be a random sample from distribution Bernoulli(p). Suppose we want to construct a confidence set for $h(p) = p/(1-p)$. Denote $\hat{p} = \bar{X}_n$. Then

$$\sqrt{n}(\hat{p} - p) \Rightarrow N(0, p(1-p))$$

In addition,

$$h'(p) = \frac{(1-p) + p}{(1-p)^2} = \frac{1}{(1-p)^2}$$

By delta-method,

$$\sqrt{n}(h(\hat{p}) - h(p)) \Rightarrow N(0, p/(1-p)^3)$$

So, $\hat{V}(h(\hat{p})) = \hat{p}/((1-\hat{p})^3 n)$. Thus, a confidence interval for $p/(1-p)$ is

$$\left[\frac{\hat{p}}{1-\hat{p}} + z_{\alpha/2} \sqrt{\frac{\hat{p}}{(1-\hat{p})^3 n}}, \frac{\hat{p}}{1-\hat{p}} + z_{1-\alpha/2} \sqrt{\frac{\hat{p}}{(1-\hat{p})^3 n}} \right].$$

4.1 Confidence Sets Based on LM and LR Tests

In addition to the Wald statistic, we can invert tests based on the *LM* and the *LR* statistics as well. However, these confidence sets are usually more involved.

Let X_1, \dots, X_n be a random sample from the distribution Bernoulli(p). Then the joint log-likelihood is

$$l_n = \log \left(p^{\sum X_i} (1-p)^{n-\sum X_i} \right) = \sum X_i \log p + (n - \sum X_i) \log(1-p)$$

So,

$$\frac{\partial l_n}{\partial p} = \frac{\sum X_i}{p} - \frac{n - \sum X_i}{1-p}$$

and

$$I(p) = \frac{1}{p(1-p)}$$

Thus,

$$\begin{aligned}
LM &= \left(\frac{\sum X_i/p - (n - \sum X_i)/(1-p)}{\sqrt{n/(p(1-p))}} \right)^2 \\
&= \left(\frac{(1-p)\sum X_i - (n - \sum X_i)p}{\sqrt{np(1-p)}} \right)^2 \\
&= \left(\frac{\sum X_i - np}{\sqrt{np(1-p)}} \right)^2
\end{aligned}$$

We know that $LM \Rightarrow \chi_1^2$. So, the confidence set based on inverting the LM test is

$$\left\{ p \in (0, 1) : \left| \frac{\sum X_i - np}{\sqrt{np(1-p)}} \right| \leq z_{1-\alpha/2} \right\}$$

Note that it is the solution to a quadratic inequality.

As for the LR test,

$$l_n^{ur} - l_n^r = \sum X_i \log(\hat{p}/p_0) + (n - \sum X_i) \log((1 - \hat{p})/((1 - p_0)))$$

So, the confidence set based on inverting the LR test is

$$\left\{ p \in (0, 1) : 2 \left(\sum X_i \log(\hat{p}/p) + (n - \sum X_i) \log((1 - \hat{p})/((1 - p))) \right) \leq \chi_{1-\alpha}^2(1) \right\}$$

It is the solution to a nonlinear inequality.

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