Lecture 9: Quantile Methods 2

1. Equivariance.
2. GMM for Quantiles.
3. Endogenous Models
4. Empirical Examples

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1. Equivariance to Monotone Transformations.

Theorem (Equivariance of Quantiles under Monotone Transformations): Let $Y = Q(U|X)$ be the Skorohod representation of Y in terms of its conditional quantile function, then for the weakly increasing transformation $T(\cdot|X)$, which is possibly X-dependent, we have

$$T(Y|X) \equiv T[Q(U|X)|X] \equiv Q_{T(Y|X)}(U|X).$$

Proof: The composition $T[Q[\cdot|X]|X]$ of two monotone weakly increasing functions gives a weakly increasing monotone function with domain $[0,1]$. Thus the composition is the proper quantile function. $\square$

Remark. The same is generally not true of the conditional expectation function:

$$E[T(Y|X)|X] \neq T[E(Y|X)|X].$$

Affine or location-scale transformations commute with conditional expectations, but more general transformations usually do not.

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Examples.

1. $Y = \ln W$ and $Q_{Y|X}(u) = X'\beta(u)$, then

$$Q_W(u|X) = \exp(X'\beta(u)).$$

The same cannot be done *generally* for mean regression.

Many standard duration models specify

$$\ln W = x'\beta + \epsilon, \quad \epsilon \text{ is indep of } X$$

where W is a positive random variable (duration, capital stock in (S,s) models, wage). Quantile regression allows us to cover and immediately generalize these models.

2. $Y^b = \max[0, Y]$ and $Q_{Y|X}(u) = x'\beta(u)$, then

$$Q_{Y^b}[u|X] = \max[0, x'\beta(u)].$$

This is censored quantile regression model. Estimation can be done using nonlinear quantile regression (Powell, 1984, JoE). A computationally attractive approximation to Powell's estimator is given in Chernozhukov and Hong (2002, JASA).

3. $Y^c = 1\{Y > 0\}$ and $Q_{Y|X}(u) = x'\beta(u)$, then

$$Q_{Y^c}[u|X] = 1\{x'\beta(u) > 0\}.$$

This is the binary quantile regression or maximum score model. Estimation can be done using nonlinear quantile regression, known as the maximum score estimator (Manski, 1975, Horowitz, 1997, Kordas, 2005). This estimator is very hard to compute.

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GMM and Bayesian GMM for Quantiles, cf. Chernozhukov and Hong (JoE, 2003)

Moment restriction:

$$E[u - 1(Y < X'\beta(u))|Z] = 0,$$

where Z could differ from X for endogenous cases.

Sample analogs of these equations are piecewise-constant functions.

$$\widehat{Q}(\beta) = n\frac{1}{2}\widehat{g}(\beta)^T W_n \widehat{g}(\beta)$$

with

$$\widehat{g}(\beta) = E_n[(u - 1(Y_i \leq X_i'\beta)\big) Z_i]$$

and

$$W_n = \frac{1}{u(1-u)}\left[E_n[Z_i Z_i']\right]^{-1}.$$

Conventional extremum estimator is

$$\beta^* = \arg\inf_{\beta\in\Theta} \widehat{Q}(\beta),$$

but it is hard, if not impossible, to compute, except on paper.

Chernozhukov and Hong (JoE, 2003) propose to use quasi-posterior mean ("Bayesian GMM"):

$$\widehat{\beta} = \int_\Theta \beta \left[\frac{e^{-\widehat{Q}(\beta)}}{\int_\Theta e^{-\widehat{Q}(\beta')} d\beta'}\right] d\beta.$$

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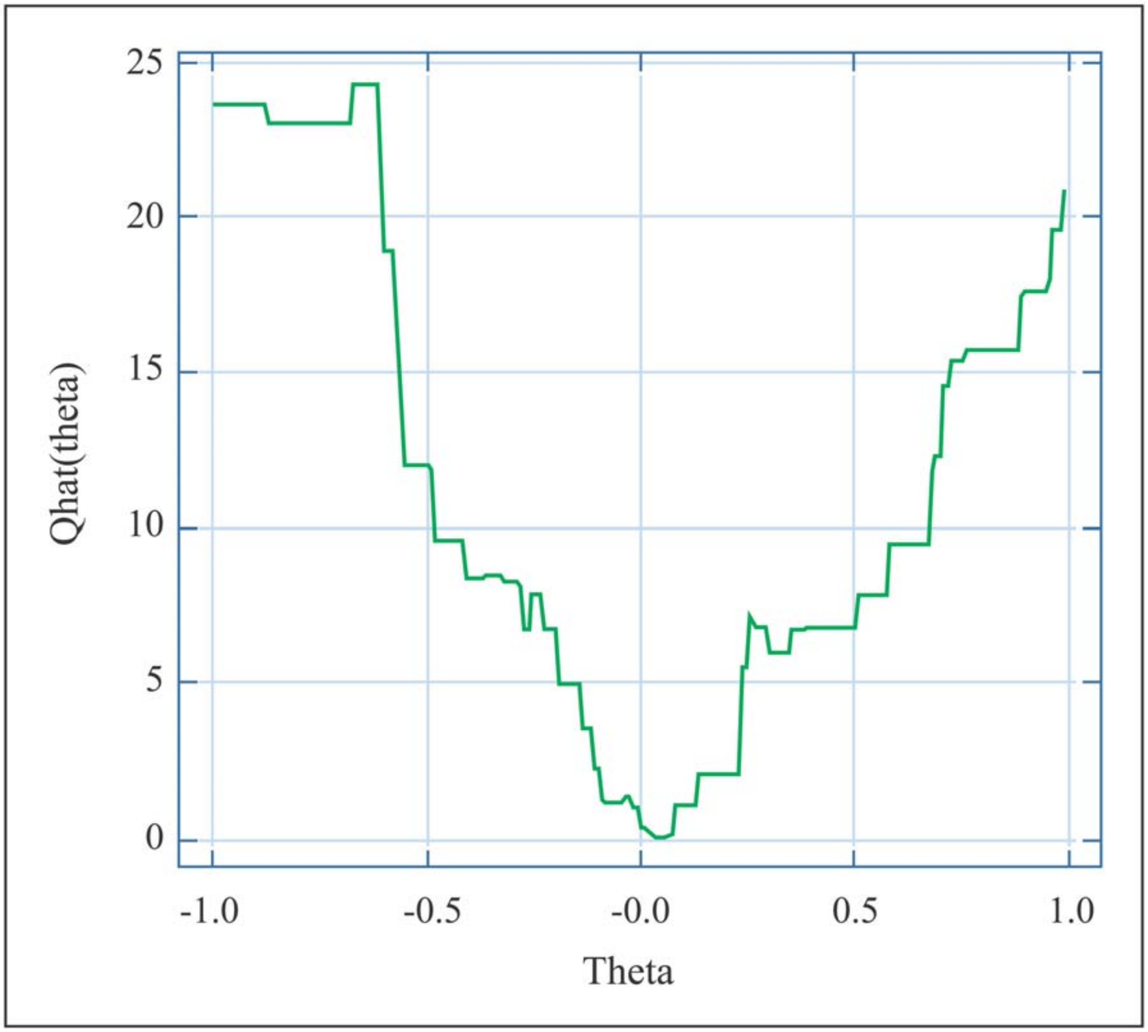
Integration via Markov Chain Monte Carlo.



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Theorem (Chernozhukov and Hong, 2003): Under regularity conditions, quasi-posterior mean and extremum estimators are first-order equivalent:

$$\sqrt{n}(\widehat{\beta} - \beta(u)) = \sqrt{n}(\beta^{*} - \beta(u)) + o_p(1)$$
$$= (G'WG)^{-1}G'W\sqrt{n}\widehat{g}(\widehat{\beta}(u)) + o_p(1)$$
$$\xrightarrow{d} N(0, (G'WG)^{-1}),$$

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where

$$G = \nabla_\beta E[(u - 1(Y_i \leq X_i'\beta)\big)Z_i] = Ef_Y(X'\beta(u)|X, Z)ZX',$$

and

$$W = \frac{1}{u(1-u)} \left(EZZ'\right)^{-1}.$$

Remarks:

1) Estimation of G can be done analogously to estimation of J in QR case.

2) For practical experience with computation, see Chernozhukov and Hong (2003).

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2. An Endogenous Quantile Regression Model

Reference: Chernozhukov and Hansen (2004, 2005, 2006), Chernozhukov and Hong (2003).

Empirical Applications: Hausman and Sidak (2003), Januscewski (2003), others.

Model:

$$
\begin{aligned}
&Y = D'\alpha(U) + X'\beta(U),\\
&D = \delta(X, Z, V),\\
&U|Z, X \sim \text{Uniform}(0, 1),\\
&u \mapsto D'\alpha(u) + X'\beta(u) \text{ strictly increasing in } u.
\end{aligned}
$$

- $D'\alpha(u) + X'\beta(u)$ is Structural Quantile Function/ Quantile Treatment Response Function

  Y is the outcome variable,
  X covariates,
  D endogenous variables,
  Z instruments,
  U outcome disturbance,
  V selection disturbance.

- Dependence of U and V causes endogeneity.

- **Independence** of U and Z is **crucial**.

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Example: Demand-Supply

$$
\begin{aligned}
&\ln Q_p = \alpha_0(U) + \alpha_1(U) \cdot \ln(p), \\
&\ln S_p = f(p, Z, \mathcal{U}), \\
&P \in \{p : Q_p = S_p\}, \\
&U \text{ independent of } Z \text{ and normalized } U(0,1).
\end{aligned} \tag{1}
$$

U is a demand disturbance, "level of demand"; $\mathcal{U}$ is a supply disturbance, "level of supply";

$p \mapsto \alpha_0(1/2)+\alpha_1(1/2)\ln(p)$ is the median demand curve;

$p \mapsto \alpha_0(u) + \alpha_1(u)\ln(p)$ is the u-quantile demand curve;

demand elasticity $\alpha_1(u)$ varies with u.

Equilibrium quantity Y and price P will satisfy

$$
\begin{aligned}
&\ln Y = \alpha_0(U) + \alpha_1(U) \ln P, \\
&P = \delta(Z, \underbrace{U, \mathcal{U}, \text{``}sunspots\text{''}}_{V}), \\
&U \quad \text{is independent of} \quad Z.
\end{aligned}
$$

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Example: A Roy Type Model for a Training Program

Potential/Latent earning outcomes Y_d in two training states d:

- (Y_0, Y_1)
- $Y_d = q(d, U_d), \quad d \in \{0, 1\}, \quad U_d \sim U(0, 1)$

- $q(d, u) = \alpha(u) + \delta(u)d$, $\quad \delta(u)$ is QTE
- $D = \arg\max_{d \in \{0,1\}} E[Utility(q(d, U_d), Z) | Z, V]$
- Assume (U_0, U_1) are independent of Z.

Hence,

- $Y = Y_D = q(D, U), \quad U = D \cdot U_1 + (1 - D)U_0,$
- $D = \delta(Z, V),$
- but U is not necessarily independent of Z.

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Remark.* Sufficient conditions for independence of U from Z:

In order to derive independence of realized U from Z, we need either of the following assumptions in addition to (U_0, U_1) being independent of Z:

I. **Rank Invariance:** $U = U_0 = U_1$. Same Rank Across Training States.

II. **Rank Similarity:** $U_0 =_d U_1 \; | Z, V$. Noisy Slippage of Ranks Across Training States;

e.g. $U_d = f(V + \eta_d), \quad \eta_d$ are i.i.d.

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3. Estimation

Main Statistical Implication is a moment restriction:

$$P[Y \leq D'\alpha(u) + X'\beta(u)|Z, X] = u \quad (*)$$

This follows from independence and

$$\{Y \leq D'\alpha(u) + X'\beta(u)\} \Leftrightarrow \{D'\alpha(U) + X'\beta(U) \leq D'\alpha(u) + X'\beta(u)\}$$
$$\Leftrightarrow \{U \leq u\}.$$

Estimation:

Approach 1: GMM, see the previous discussion.

Approach 2: Inverse Quantile Regression, is a modification of conventional QR

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INVERSE QUANTILE REGRESSION*

Principle: Find $D'\alpha$ such that Quantile Regression of $Y - D'\alpha$ on X and Z returns **0** as the estimate of coefficients on Z.

1. Given $\alpha \in \mathcal{A}$, run QR of $Y - D'\alpha$ on X and Z:

$$(\widehat{\beta}(\alpha), \widehat{\gamma}(\alpha)) \equiv \arg\min_{(\beta,\gamma)} \Big[E_n \rho_u (Y - D'\alpha - X'\beta - Z'\gamma) \Big],$$

2. Pick α such that the $Wald$ statistic for testing the exclusion of Z is as small as possible:

$$\widehat{\alpha}(u) \in \arg\inf_{\alpha \in \mathcal{A}} W_n[\alpha],$$
$$W_n[\alpha] = n\Big(\widehat{\gamma}(\alpha)'\widehat{\Omega}_{\gamma}^{-1}[\alpha]\widehat{\gamma}(\alpha)\Big)$$

Estimate $\widehat{\beta}(u)$ by $\widehat{\beta}(\widehat{\alpha})$.

In practice:

1) given sequence $\alpha_j, j = 1, ..., J$,
run J Quantile Regressions of $Y - D'\alpha_j$ on X and Z,

2) pick $\widehat{\alpha}(u)$ as the value of α_j that minimizes $W_n[\alpha_j]$ – the testing statistic for excluding the instruments.

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Asymptotics of IQR. The estimator is consistent and asymptotically normal, cf. Chernozhukov and Hansen (2001) for details.

Computing. This method works very well if endogenous regressors D are one- or two-dimensional and X could be very high-dimensional. When D is high-dimensional, the previous quasi-Bayesian GMM approach becomes preferable.

Robust Confidence Region. In addition to create robust confidence region for α can collect all α_j's s.t.

$$W_n[\alpha_j] \leq c_p,$$

where c_p is the critical value for the Wald statistic. The region is a (numerical approximation) to a confidence region that contains $\alpha(u)$ with probability p, cf. Chernozhukov and Hansen (2008, JoE) for details.

This second method is robust to partial identification and weak identification.

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4. Empirical Applications

Demand Estimation

Data on prices P and quantities Q at Fulton Fish market, which is for Whiting fish.

The instruments Z are supply shifters such as weather conditions at sea.

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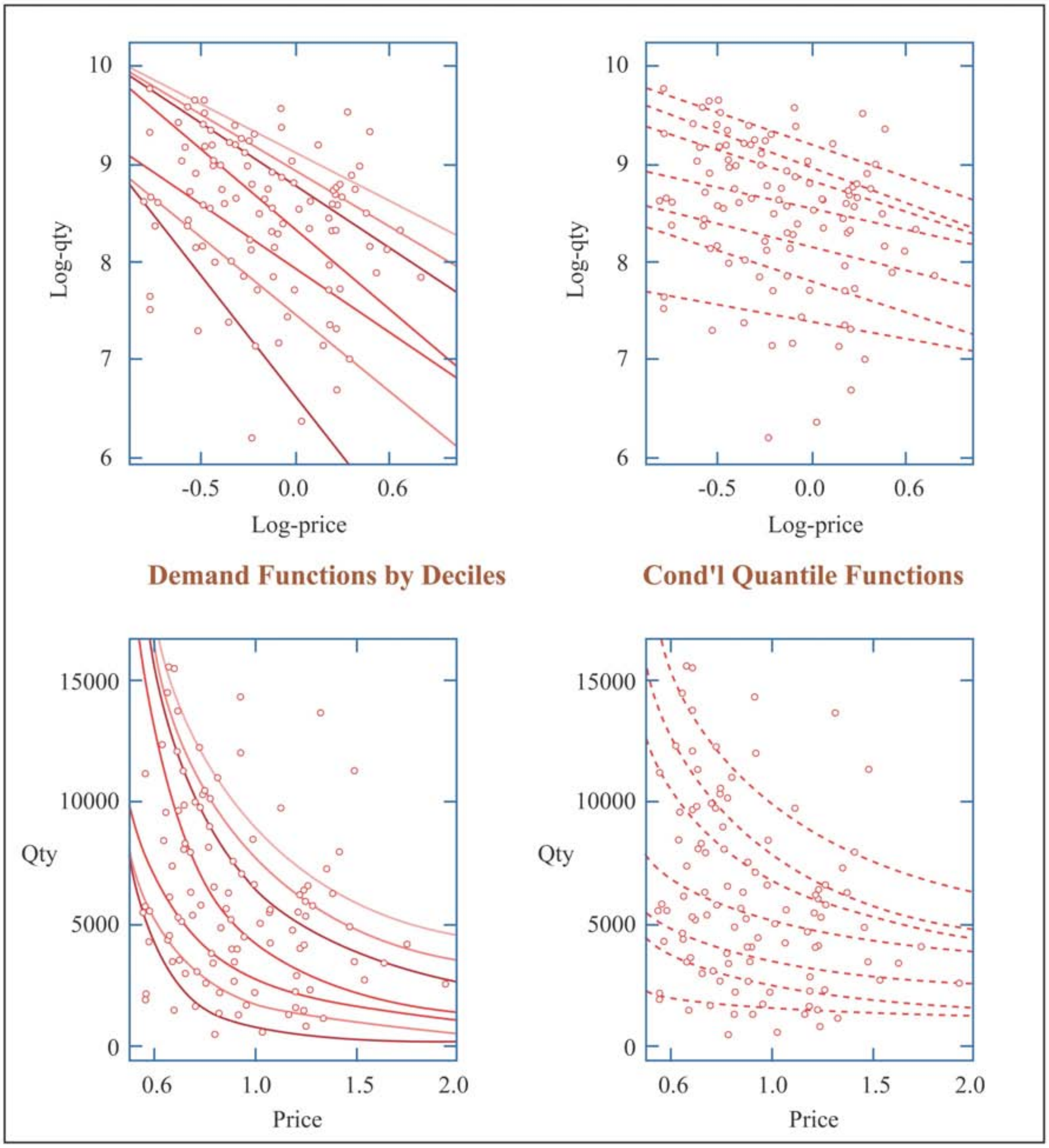

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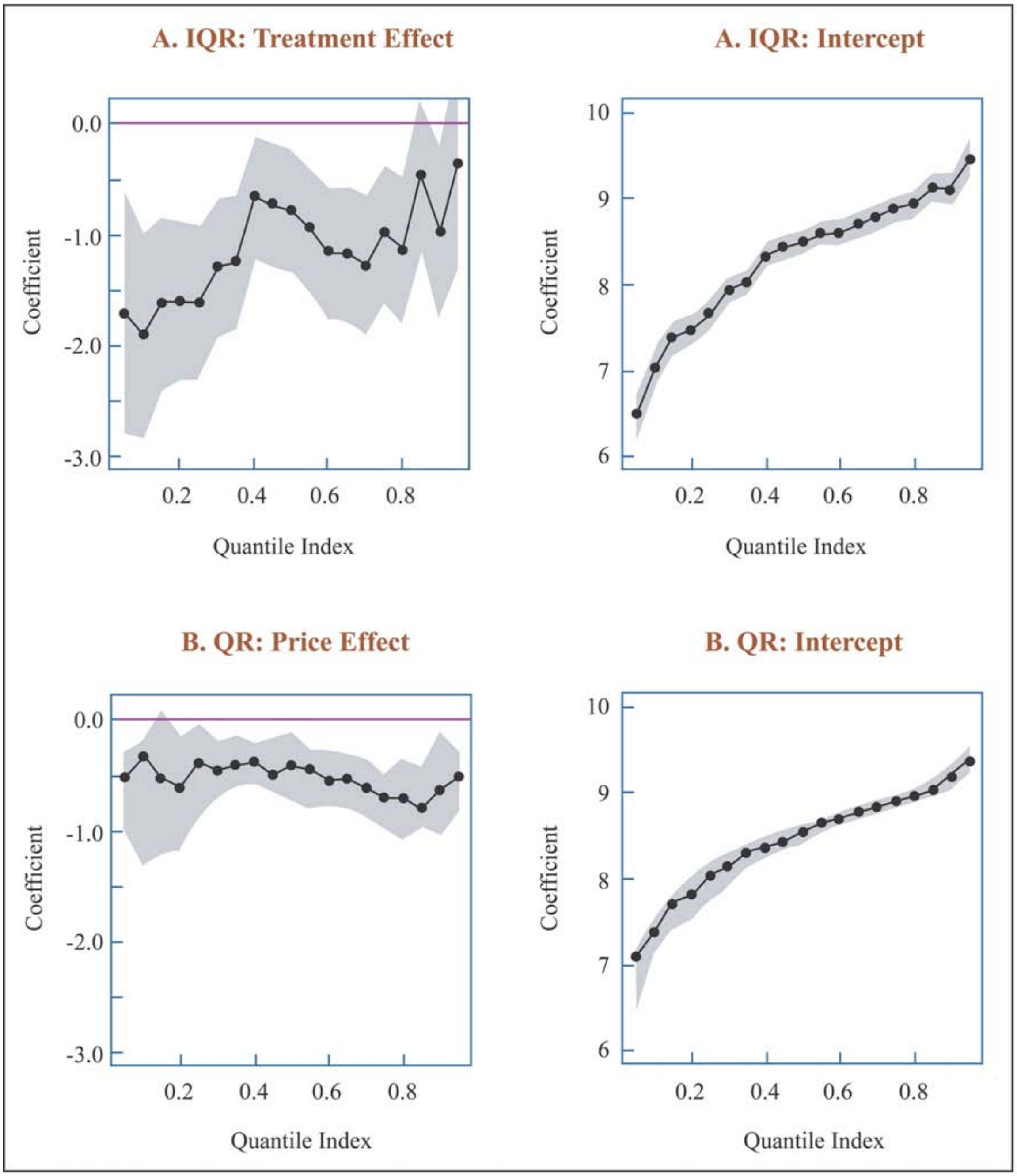

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Evaluation of JTPA

JTPA: descussed in detail in Heckman and Smith (1997), Abadie et al. (2003)

Sample based on a randomized experiment with imperfect compliance.

Men,
$Y =$ 30 Month Earnings,
$D =$ Participation,
$Z =$ Offer of Participation,
$X =$ Good Set of Controls (15).

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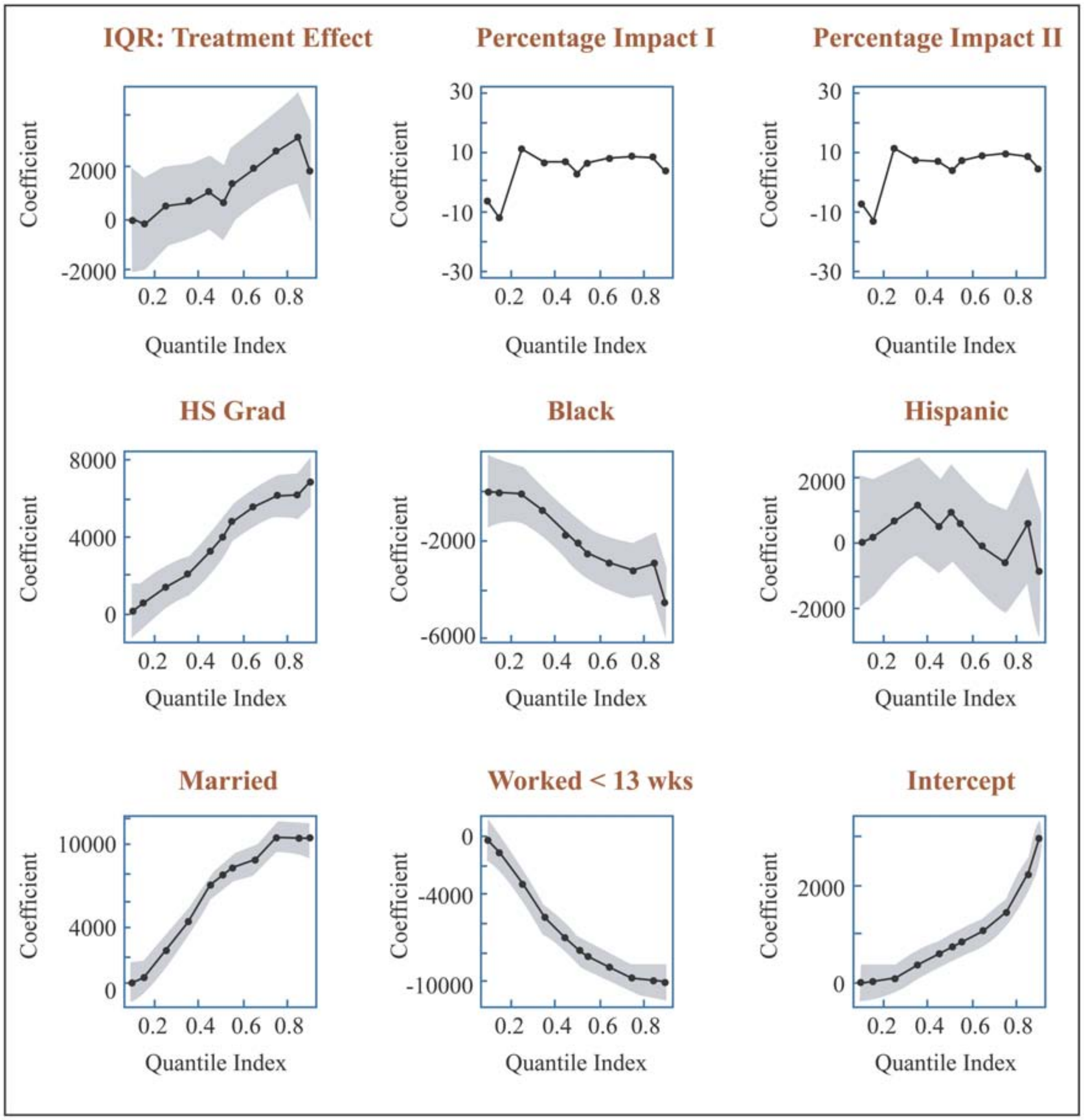
Evaluation of JTPA



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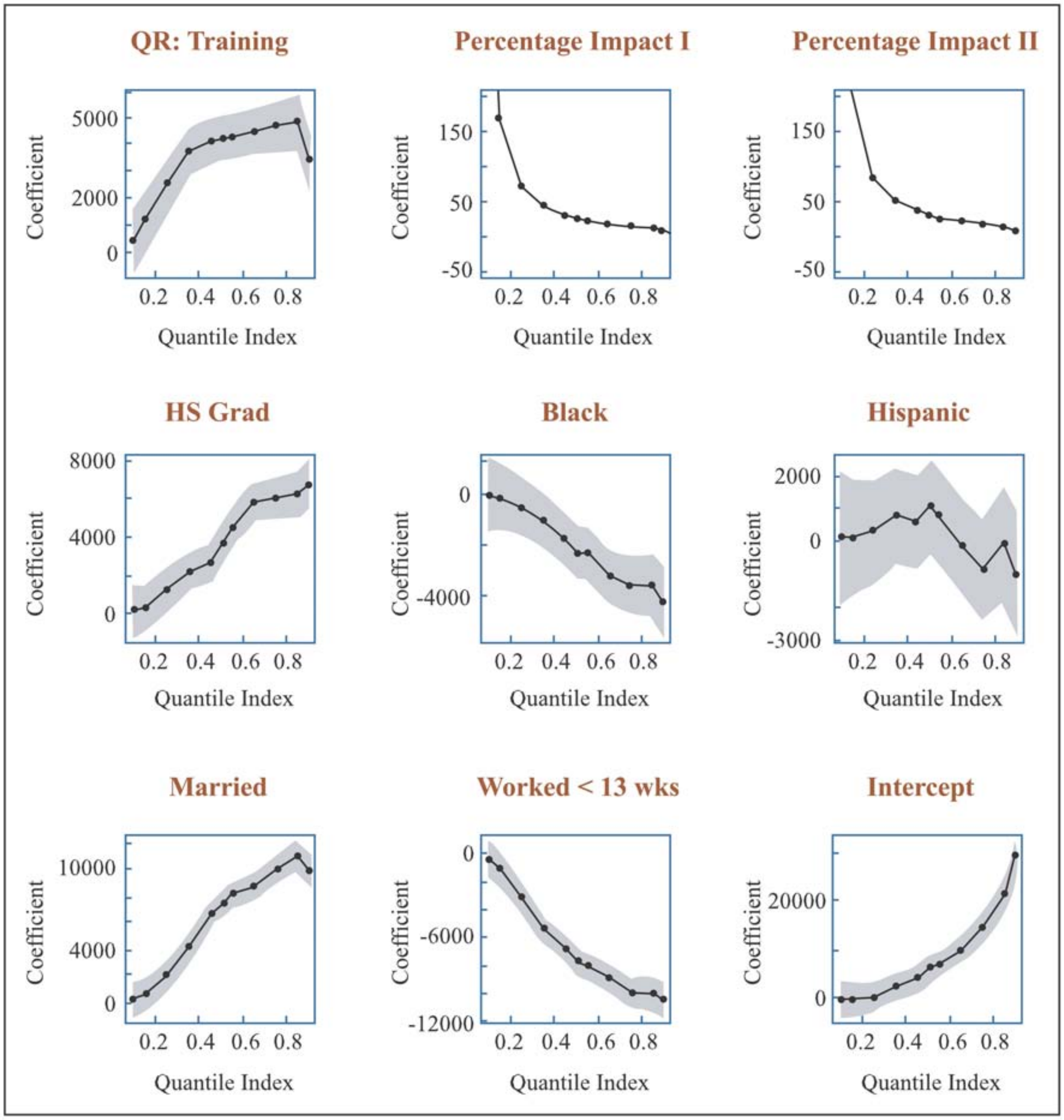

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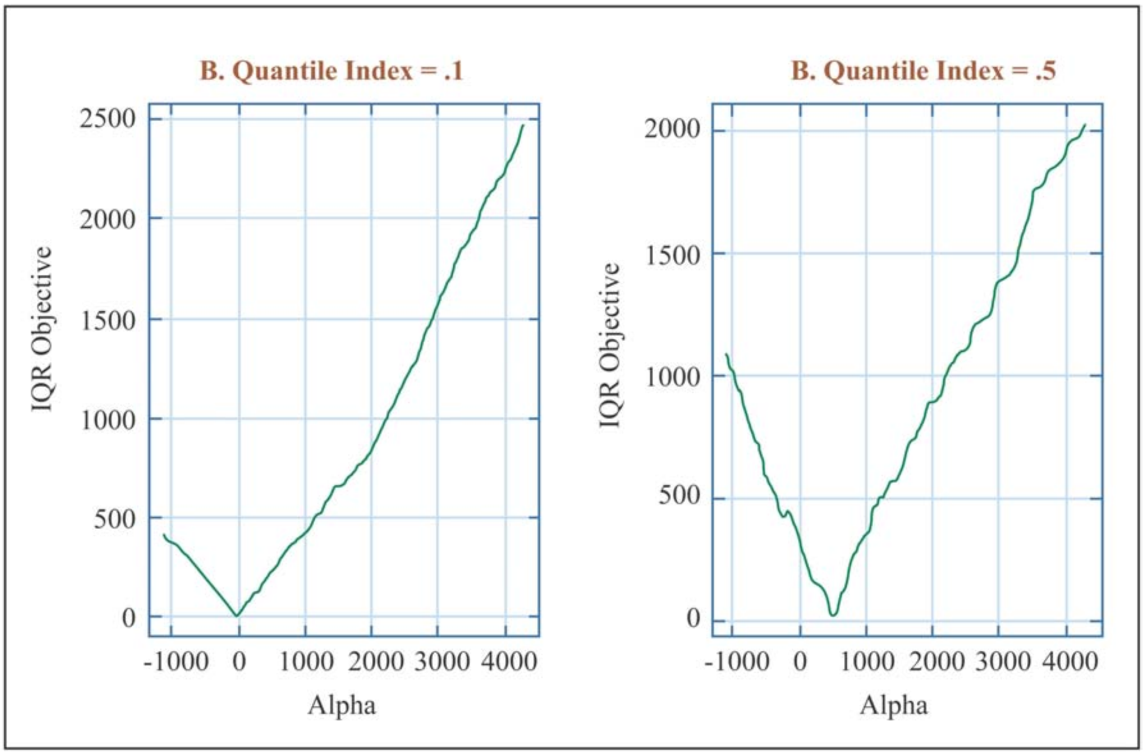

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