14.451 Lecture Notes 3

Guido Lorenzoni

Fall 2009

1 Showing that T is a contraction

1.1 Blackwell's sufficient conditions

One important step in applying our argument to the map T is to show that T is a contraction.

Let $C(X)$ be the space of bounded functions on X let $\|.\|$ be the sup norm.

Take any map

$$T : C(X) \rightarrow C(X).$$

(This does not need to come from any optimization problem.)

Assume:

1. T is monotone, if $f, g \in C(X)$ and $f(x) \geq g(x)$ for all $x \in X$, then $Tf(x) \geq Tg(x)$ for all $x \in X$.
2. T satisfies a "discounting" property: there is a $\delta \in (0,1)$ such that for any $a \geq 0$ and any $g \in C(X)$ the function $f(x) = g(x) + a$ satisfies

$$T(f(x) - g(x)) \leq \delta a.$$

Then T is a contraction.

To prove this let

$$a = \sup_{x \in X} |f(x) - g(x)| = \|f - g\|.$$

Suppose without loss of generality that

$$\sup_{x \in X} |Tf(x) - Tg(x)| = \sup_{x \in X} (Tf(x) - Tg(x))$$

(if this does not hold then it must be

$$\sup_{x \in X} |Tf(x) - Tg(x)| = \sup_{x \in X} (Tg(x) - Tf(x))$$

and the same argument works with the roles of f and g reversed). Then let $h(x) = g(x) + a$. We have $h(x) \geq f(x)$ for all x by definition. So

$$Tf(x) - Tg(x) \leq Th(x) - Tg(x)$$

from monotonicity and

$$Th(x) - Tg(x) \leq \delta a$$

from discounting. Combining them we have

$$Tf(x) - Tg(x) \leq \delta a$$

which implies

$$\|Tf - Tg\| = \sup_{x \in X} |Tf(x) - Tg(x)| = \sup_{x \in X} (Tf(x) - Tg(x)) \leq \delta a = \delta \|f - g\|.$$

1.2 Applying Blackwell's conditions

Now we go back to our dynamic programming problem and show that T, defined as

$$Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y)$$

is indeed a contraction. (Here we assume we already know that T maps bounded continuous functions into bounded continuous functions).

To apply Blackwell's theorem we need to check conditions 1 and 2.

1. To see that T is monotone suppose $f \geq g$. Take any $x \in X$ and suppose

$$y' \in \arg \max_{y \in \Gamma(x)} F(x, y) + \beta g(y).$$

Then

$$\begin{aligned} Tf(x) &= \max_{y \in \Gamma(x)} F(x, y) + \beta f(y) \geq \\ &\geq F(x, y') + \beta f(y') \geq \\ &\geq F(x, y') + \beta g(y') = Tg(x). \end{aligned}$$

Since this holds for all $x \in X$, we are done.

2. So see that T satisfies discounting notice that for any $a \geq 0$ is $f(x) = g(x) + a$

$$\begin{aligned} Tf(x) &= \max_{y \in \Gamma(x)} F(x, y) + \beta (g(y) + a) = \\ &= \left\{ \max_{y \in \Gamma(x)} F(x, y) + \beta g(y) \right\} + \beta a \\ &= Tg(x) + \beta a. \end{aligned}$$

So discounting applies (with $\delta = \beta$).

2 Inductive arguments

Using induction we can prove properties of the value function.

The general idea is to use the fact that our fixed point V is the limit of $T^n f_0$. So if we start from a

$$f_0 \in D$$

(f_0 satisfies D) and can prove that

$$Tf \in D \text{ if } f \in D$$

(T preserves property D), then provided that D is a closed subset of our original metric space $C(X)$ then

$$V \in D.$$

2.1 Proving that V is monotone

Make all assumptions of bounded dynamic programming plus:

- $F(x, y)$ is increasing in its first argument;
- $\Gamma(x)$ is monotone in the sense that

$$\Gamma(x') \subset \Gamma(x'') \text{ if } x'' \geq x'.$$

Then $V(x)$ is increasing in x.

Proof. We need to prove our induction step:

$$Tf \text{ is increasing if } f \text{ is increasing.}$$

We actually prove a stronger version:

$$Tf \text{ is increasing if } f \text{ is non-decreasing.} \tag{1}$$

Pick an $x', x'' \in X$ with $x'' \geq x'$ (with at least one $>$). Choose a

$$y' \in \arg\max_{y \in \Gamma(x')} F(x', y) + \beta f(y).$$

Then $y' \in \Gamma(x'')$ by monotonicity of Γ, so

$$\begin{aligned} Tf(x'') &= \max_{y \in \Gamma(x'')} F(x'', y) + \beta f(y) \geq F(x'', y') + \beta f(y) > \\ &> F(x', y') + \beta f(y) = Tf(x'), \end{aligned}$$

where the last inequality comes from the fact that F is increasing. The space of strictly increasing functions is not closed, but the space of non-decreasing functions is closed (and is the closure of the space of increasing functions). So since V is the limit of $T^n f_0$, we have V non-decreasing. Moreover

$$V = TV$$

and V is non-decreasing. So (1) implies that V is increasing. QED.

2.2 Proving that V is concave

Same idea: move in the space of concave functions.

Now the extra assumptions we need are:

- $F(x, y)$ is concave;
- Γ is a convex in the sense that if $y' \in \Gamma(x')$ and $y'' \in \Gamma(x'')$ then

$$\alpha y' + (1 - \alpha) y'' \in \Gamma(\alpha x' + (1 - \alpha) x'') \text{ for all } \alpha \in [0, 1].$$

Then $V(x)$ is concave.

Proof. Again we need our inductive step. Suppose $f(x)$ is concave. Take any $x', x'' \in X$ and

$$\begin{aligned} y' &\in \arg \max_{y \in \Gamma(x')} F(x', y) + \beta f(y) \\ y'' &\in \arg \max_{y \in \Gamma(x'')} F(x'', y) + \beta f(y). \end{aligned}$$

Take any $\alpha \in [0, 1]$ and let $x''' = \alpha x' + (1 - \alpha) x''$. Then

$$y''' = \alpha y' + (1 - \alpha) y'' \in \Gamma(x''')$$

by convexity of Γ. So

$$\begin{aligned} &\max_{y \in \Gamma(x''')} F(x''', y) + \beta f(y) \geq F(x''', y''') + \beta f(y''') \geq \\ &\alpha (F(x', y') + \beta f(y')) + (1 - \alpha)(F(x'', y'') + \beta f(y'')) \end{aligned}$$

where the last inequality follows from the concavity of F and f. So we have

$$Tf(x''') \geq \alpha Tf(x') + (1 - \alpha) Tf(x''),$$

showing that Tf is concave. Since the space of concave functions is closed, we can start at any f_0 concave and we end up at V concave. Again, if needed we can strengthen to strict concavity by making an extra step (like going from weak to strong monotonicity). QED

14.451 Dynamic Optimization Methods with Applications
Fall 2009