14.451 Lecture Notes 4

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1 Continuity of the policy function

We are in the case of bounded returns and we are assuming strict concavity of F and convexity of Γ.

We proved that V is strictly concave. This has one important implication about the policy:

The policy correspondence $G(x)$ is single valued, i.e., it is a function. We will use $g(x)$ to denote it.

To prove that $G(x)$ is single-valued we use (1) the thm of the maximum, which shows that G is u.h.c. and (2) the fact that for single-valued correspondences u.h.c.$\rightarrow$ continuity.

1. It is useful to recall steps from thm of the maximum. To apply that theorem we only use the facts that (a) $F(x,y)+\beta V(y)$ is continuous (by Thm 4.6) and (b) $\Gamma(x)$ is continuous (by assumption).

   Then if $x_n \rightarrow x$ and $y_n \in G(x_n)$ we want to find a convergent subsequence with $y_{n_k} \rightarrow y \in G(x)$. Since X is compact, a convergent subsequence $\{y_{n_k}\}$ must exist. Take any $y' \in \Gamma(x)$. Then since Γ is l.h.c. there must be a sequence $\left\{y'_{n_k}\right\}_{k=K}^{\infty} \rightarrow y$ with $y_{n_k} \in \Gamma(x_{n_k})$. But then

   $$F(x_{n_k}, y_{n_k}) + \beta V(y_{n_k}) \geq F\left(x_{n_k}, y'_{n_k}\right) + \beta V\left(y'_{n_k}\right) \text{ for all } k \geq K$$

   and taking lims we have

   $$F(x,y) + \beta V(y) \geq F(x,y') + \beta V(y')$$

   since this is true for all y' we have $y \in G(x)$.

2. Next we want to show that u.h.c. and single valued yield continuity. Suppose $x_n \rightarrow x$ and $\|y_{n_k} - y\| > \delta$ for all k for some $\delta > 0$ for some subsequence $\{y_{n_k}\}$. But then take the sequence $\{x_{n_k}\}$. By (1) there must be a subsequence of $\{y_{n_k}\}$ that converges to some $y' \in G(x)$. Since G is single-valued we must have $y' = y$ and since $\|y_{n_k} - y\| > \delta$ for all k we have a contradiction.

2 Differentiability of the value function

To characterize the optimum sometimes it is useful to look at the first order condition of the problem in FE:

$$F_y(x,y) + \beta V'(y) = 0.$$

Clearly to do so we need F to be differentiable (in its second argument), but we also need V to be differentiable. What do we know about the differentiability of V?

Example

Simple finite horizon problem (really 1 period!):

$$V(x) = \max_{y \in [0,1]} y^2 - xy$$

where the initial state is $x \geq 0$.

If $x \leq 1$ we have $V(x) = 1 - x$ if $x > 1$ $V(x) = 0$.

Value function is not differentiable at $x = 1$, why? Lack of concavity$\rightarrow$ discontinuity in the policy function.

So we hope that concavity can give us continuity of the policy can also give us differentiability.

Fact 1. If a function $f : X \rightarrow R$ is concave (with X convex subset of R^n) the function f admits a subgradient $p \in R^n$, i.e. a p such that

$$f(x) - f(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in X.$$

Notice:

- This fact is true whether or not f is differentiable.
- If f is differentiable then p is unique and is the gradient of f at x_0.

The converse of the second point is also true:

Fact 2. If f is concave and has a unique subgradient, then f is differentiable.

We can now prove differentiability of V

Theorem 1 *Suppose F is differentiable in x and the value function is concave, $x_0 \in intX$ and $y_0 = g(x_0) \in int\Gamma(x_0)$, then V is differentiable at x_0 with*

$$\nabla V(x_0) = \nabla F_x(x_0, y_0)$$

Proof. The idea is to find a concave function $W(x)$ that is a lower approximation for $V(x)$ in a neighborhood of x_0 and that is differentiable. Let us use

$$W(x) = F(x, y_0) + \beta V(y_0)$$

Given continuity of Γ and the fact that $y_0 \in int\Gamma(x_0)$, we can find a neighborhood D of x_0 such that $y_0 \in \Gamma(x)$ for all $x \in D$. Then

$$W(x) \leq V(x) \text{ for all } x \in D$$

and

$$W(x_0) \leq V(x_0).$$

Since V is concave it has a subgradient p and we have the chain of inequalities

$$W(x) - W(x_0) \leq V(x) - V(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in D.$$

Since W is differentiable in x it must be

$$p = \nabla W(x_0)$$

so the subgradient is unique, and, by fact 2, V is differentiable. ■

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