14.451 Lecture Notes 6

Guido Lorenzoni

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1 Euler equations

Consider a sequence problem with F continuous differentiable, strictly concave increasing in its first l arguments ($F_x \geq 0$). Suppose the state x_t is a non-negative vectors ($X \subset R^l_+$).

Then we can use the Euler equation and a transversality condition to find an optimum.

If a sequence $\{x^*_t\}$ satisfies $x^*_{t+1} \in int\Gamma(x^*_t)$ and

$$F_y\left(x^*_t, x^*_{t+1}\right) + \beta F_x\left(x^*_{t+1}, x^*_{t+2}\right) = 0 \tag{1}$$

for all t, and the additional condition

$$\lim_{t\to\infty} \beta^t F_x\left(x^*_t, x^*_{t+1}\right) x^*_t = 0 \tag{2}$$

then the sequence is optimal.

To prove it we first use concavity to show that for any feasible sequence $\{x_t\}$ we have

$$F(x_t, x_{t+1}) \leq F\left(x^*_t, x^*_{t+1}\right) + F_x\left(x^*_t, x^*_{t+1}\right)(x_t - x^*_t) + F_y\left(x^*_t, x^*_{t+1}\right)\left(x_{t+1} - x^*_{t+1}\right)$$

summing term by term for $t = 1, ..., T$ (discounting each term by β^t) yields

$$\begin{aligned}
\sum_{t=0}^T \beta^t F(x_t, x_{t+1}) &\leq \sum_{t=0}^T \beta^t \left(F\left(x^*_t, x^*_{t+1}\right) + F_x\left(x^*_t, x^*_{t+1}\right)(x_t - x^*_t) + F_y\left(x^*_t, x^*_{t+1}\right)\left(x_{t+1} - x^*_{t+1}\right)\right) \\
&= \sum_{t=0}^T \beta^t F\left(x^*_t, x^*_{t+1}\right) + \beta^T F_y\left(x^*_T, x^*_{T+1}\right)\left(x_{T+1} - x^*_{T+1}\right) \\
&= \sum_{t=0}^T \beta^t F\left(x^*_t, x^*_{t+1}\right) + \beta^{T+1} F_y\left(x^*_{T+1}, x^*_{T+2}\right)\left(x^*_{T+1} - x_{T+1}\right) \\
&\leq \sum_{t=0}^T \beta^t F\left(x^*_t, x^*_{t+1}\right) + \beta^{T+1} F_x\left(x^*_{T+1}, x^*_{T+2}\right) x^*_{T+1}
\end{aligned}$$

The second line follows from the fact that all the terms $\beta^t F_y\left(x^*_t, x^*_{t+1}\right)\left(x_{t+1} - x^*_{t+1}\right)$ and $\beta^{t+1} F_x\left(x^*_{t+1}, x^*_{t+2}\right)\left(x_{t+1} - x^*_{t+1}\right)$ cancel each other, by (1), and that $x_0 =$

x_0^* from feasibility. The third line follows from applying (1) one more time. The last line follows from $x_{T+1} \geq 0$ and $F_x \geq 0$.

Taking limits on both sides and using (2) shows that

$$\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \leq \sum_{t=0}^{\infty} \beta^t F\left(x_t^*, x_{t+1}^*\right).$$

2 Local stability

We are now going to use conditions (1) and (2) to characterize optimal dynamics around a steady state.

Suppose we find an x^* such that $x^* \in int\Gamma(x^*)$ and

$$F_y(x^*, x^*) + \beta F_x(x^*, x^*) = 0$$

then x^* is a steady state, i.e. $x^* = g(x^*)$ (can you prove it?)

Suppose first that the problem is quadratic: $F(x, y)$ is a quadratic, strictly concave function. So its derivatives are linear functions.

$$\begin{aligned} F_y(x_t, x_{t+1}) &= F_y(x^*, x^*) + F_{yx} \cdot (x_t - x^*) + F_{yy} \cdot (x_{t+1} - x^*) \\ F_x(x_{t+1}, x_{t+2}) &= \beta F_y(x^*, x^*) + \beta F_{xx} \cdot (x_{t+1} - x^*) + \beta F_{xy} \cdot (x_{t+2} - x^*) \end{aligned}$$

$$F_{yx} z_t + F_{yy} z_{t+1} + \beta F_{xx} z_{t+1} + \beta F_{xy} z_{t+2} = 0$$

Assumption. *The matrices F_{xy} and $F_{yx} + F_{yy} + \beta F_{xx} + \beta F_{xy}$ are non-singular.*

Then we have the 2nd order difference equation

$$z_{t+2} = \beta^{-1} F_{xy}^{-1} (F_{yy} + \beta F_{xx}) z_{t+1} + \beta^{-1} F_{xy}^{-1} F_{yx} z_t \tag{3}$$

We want to characterize the optimal dynamics using (1) and (2) as sufficient conditions. So we ask the question: "given any $z_0 = x_0 - x^*$ can we find a $z_1 = x_1 - x^*$ such that the sequence $\{z_t\}$ satisfies (3) with initial conditions (z_0, z_1) and $\lim_{t\to\infty} z_t = 0$?"

If we find such a z_1 then $x_1 = x^* + z_1$ must be equal to the optimal policy $g(x_0)$ because the sequence $\{x^* + z_t\}_{t=0}^{\infty}$ satisfies the sufficient conditions for an optimum (1) and (2). Moreover, since the problem is strictly concave x_1 must be unique.

We can restate the problem in terms of the 1st order difference equation:

$$\begin{bmatrix} z_{t+2} \\ z_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \beta^{-1} F_{xy}^{-1} (F_{yy} + \beta F_{xx}) & \beta^{-1} F_{xy}^{-1} F_{yx} \\ I & 0 \end{bmatrix}}_{M} \begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix}$$

Now we are looking for a z_1 such that

$$M^j \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} \to 0. \tag{4}$$

Can this be true for more than one z_1? No otherwise we would have multiple solutions. So the options are:

- there is a unique z_1 that satisfies (4). Then we have the policy $x_1 = g(x_0) = x^* + z_1$ and the optimal path from x_0 converges to x^*.
- there is no z_1 that satisfies (4). Then we don't have much information on $g(x_0)$ but we know that there is no optimal path starting at x_0 that converges to x^*.

We will try to find conditions so that the first option applies.

We now leave aside dynamic programming for a moment and review useful material on the general properties of difference equations.

2.1 Difference equations

General problem: characterize the limiting behavior of the sequence $Z_t = M^t Z_0$ for some square matrix M and all possible initial conditions $Z_0 \in R^{2l}$.

Useful result: given a square matrix M, it can be decomposed as

$$M = B^{-1} \Lambda B$$

where Λ is a *Jordan matrix* and B is a non-singular matrix. The elements on the diagonal of Λ are the solutions to

$$\det(\lambda I - M) = 0$$

(some of them may be complex numbers). This is called the *characteristic equation* of M and the expression on the right-hand side the *characteristic polynomial*.

Then we can analyze the dynamics of the sequence $W_t = BZ_t$. Since B is invertible, there is a one-to-one mapping between Z_t and W_t, so all the properties we can establish for $\{W_t\}$ translate into properties of $\{Z_t\}$. Convergence is much easier to analyze for the sequence W_t, because

$$W_t = BZ_t = BMu_{t-1} = BB^{-1}\Lambda Bu_{t-1} = \Lambda w_{t-1}$$

so

$$w_t = \Lambda^t w_0.$$

But the powers of a Jordan matrix Λ have nice limiting properties. The matrix is made of diagonal blocks of the form

$$\Lambda_j = \begin{bmatrix} \lambda_j & 1 & 0 & ... & 0 \\ 0 & \lambda_j & 1 & ... & 0 \\ 0 & 0 & \lambda_j & ... & 0 \\ ... & ... & ... & ... & 1 \\ 0 & 0 & 0 & 0 & \lambda_j \end{bmatrix}$$

and $\Lambda_j^t \to 0$ (a matrix of zeros) if $|\lambda_j| < 1$.

A simple example in R^2. The difference equation is

$$Z_t = MZ_{t-1},$$

with M a 2×2 matrix. Suppose M has a real eigenvalue λ with $|\lambda| < 1$ and the associated eigenvector is $\hat{Z}$. Then we have (by definition of eigenvalue and eigenvector)

$$M\hat{Z} = \lambda\hat{Z}.$$

To find $\hat{Z}$ we need the following equation

$$(\lambda I - M)\, Z = 0$$

to have a solution different from zero, but this requires $\lambda I - M$ to be non-singular, i.e. $\det(\lambda I - M) = 0$. This shows why we find the λ's by solving the characteristic equation.

Once we find λ and $\hat{Z} = [\zeta_1, \zeta_2]'$ how can we use them to solve our original problem? First remember that M has the form

$$M = \begin{bmatrix} J & K \\ I & 0 \end{bmatrix}$$

for some numbers J and K. This means that ζ_2 cannot be zero. Otherwise

$$\lambda \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} J & K \\ I & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

would give $\zeta_1 = \zeta_2/\lambda = 0$ and $\hat{Z}$ cannot be $[0,0]'$ (by definition of eigenvector).

Now take any initial condition z_0 and set $z_1 = (\zeta_1/\zeta_2)\, z_0$ so that $[z_1, z_0]'$ is proportional to $\hat{Z}$:

$$\begin{bmatrix} z_1 \\ z_0 \end{bmatrix} = \frac{z_0}{\zeta_2} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

This means that we have found a z_1 such that $[z_1, z_0]'$ is an eigenvector of M. Therefore

$$\begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix} = M^t \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} = \lambda^t \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} \to 0$$

since $|\lambda| < 1$.

In the next lecture we'll see how to generalize this (using the Jordan decomposition).

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