Chapter 1

Introduction and Growth Facts

1.1 Introduction

- In 2000, GDP per capita in the United States was $32500 (valued at 1995 $ prices). This high income level reflects a high standard of living.

- In contrast, standard of living is much lower in many other countries: $9000 in Mexico, $4000 in China, $2500 in India, and only $1000 in Nigeria (all figures adjusted for purchasing power parity).

- *How can countries with low level of GDP per person catch up with the high levels enjoyed by the United States or the G7?*

- Only by high growth rates sustained for long periods of time.

- *Small differences in growth rates over long periods of time can make huge differences in final outcomes.*

- US per-capita GDP grew by a factor ≈ 10 from 1870 to 2000: In 1995 prices, it was $3300 in 1870 and $32500 in 2000.[1] Average growth rate was $\approx 1.75\%$. If US had grown with .75% (like India,

[1] Let y_0 be the GDP per capital at year 0, y_T the GDP per capita at year T, and x the average annual growth rate over that period. Then, $y_T = (1+x)^T y_0$. Taking logs, we compute $\ln y_T - \ln y_0 = T \ln(1+x) \approx Tx$, or equivalenty $x \approx (\ln y_T - \ln y_0)/T$.

Pakistan, or the Philippines), its GDP would be only \$8700 in 1990 (i.e., $\approx 1/4$ of the actual one, similar to Mexico, less than Portugal or Greece). If US had grown with 2.75% (like Japan or Taiwan), its GDP would be \$112000 in 1990 (i.e., 3.5 times the actual one).

- At a growth rate of 1%, our children will have ≈ 1.4 our income. At a growth rate of 3%, our children will have ≈ 2.5 our income. Some East Asian countries grew by 6% over 1960-1990; this is a factor of ≈ 6 within just one generation!!!

- Once we appreciate the importance of sustained growth, the question is natural: *What can do to make growth faster?*

- Equivalently: What are the factors that explain differences in economic growth, and how can we control these factors?

- In order to prescribe policies that will promote growth, we need to understand what are the determinants of economic growth, as well as what are the effects of economic growth on social welfare. That's exactly where Growth Theory comes into picture...

1.2 The World Distribution of Income Levels and Growth Rates

- As we mentioned before, in 2000 there were many countries that had much lower standards of living than the United States. This fact reflects the high cross-country dispersion in the level of income.

- (in what follows, all figures are reproduced from Daron Acemoglu's textbook)

- **Figure 1.1** shows the distribution of (the log of) GDP per capita in 1960, 1980, and 2000 across the 147 countries in the Summers and Heston dataset.

- In 2000, the richest country was Luxembourg, with \$44000 GDP per person. The United States came second, with \$32500. The G7 and most of the OECD countries ranked in the top 25 positions, together with Singapore, Hong Kong, Taiwan, and Cyprus. Most African countries, on the other hand, fell in the bottom 25 of the distribution. Tanzania was the poorest country, with only \$570 per person–that is, less than 2% of the income in the United States or Luxemburg! In 1960, on the other hand, the richest country then was Switzerland, with \$15000; the United States was again second, with \$13000, and the poorest country was again Tanzania, with \$450.

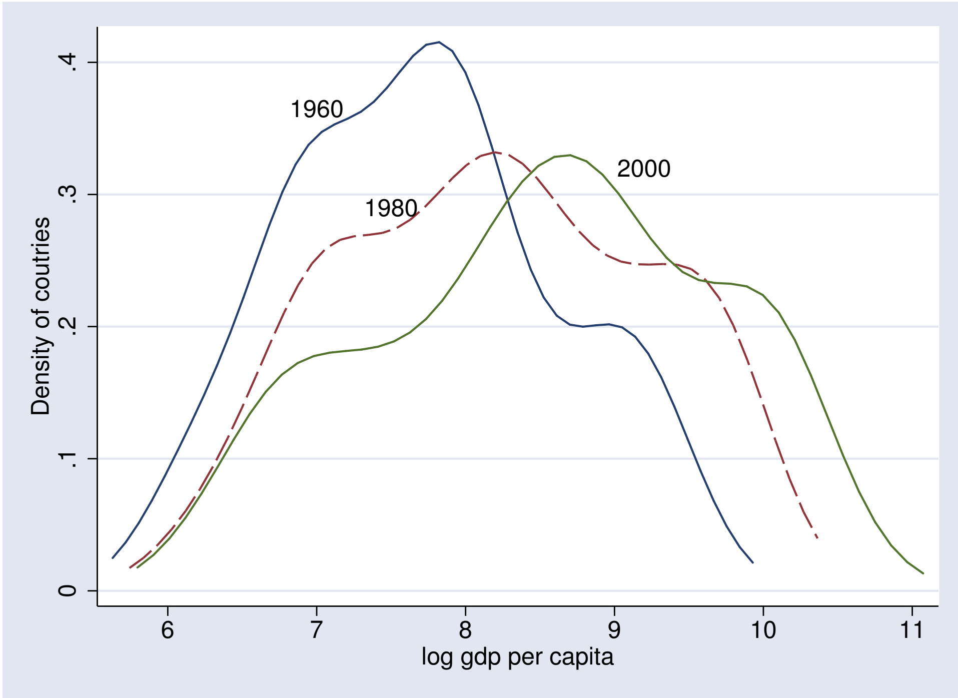

Figure 1.1: Estimates of the distribution of countries according to log GDP per capita (PPP-adjusted) in 1960, 1980 and 2000.

- The cross-country dispersion of income was thus as wide in 1960 as in 2000. Nevertheless, there were some important movements during this 40-year period. Argentina, Venezuela, Uruguay, Israel, and South Africa were in the top 25 in 1960, but none made it to the top 25 in 2000. On the other hand, China, Indonesia, Nepal, Pakistan, India, and Bangladesh grew fast enough to escape the bottom 25 between 1960 and 1970. These large movements in the distribution of income reflects sustained differences in the rate of economic growth.

- **Figure 1.2** shows the distribution of the growth rates the countries experienced between 1960 and 2000. Just as there is a great dispersion in income levels, there is a great dispersion in growth rates. The mean growth rate was 1.8% per annum; that is, the world on average was twice as rich in 2000 as in 1960. The United States did slightly better than the mean. The fastest growing country was Taiwan, with a annual rate as high as 6%, which accumulates to a factor of 10 over the 40-year period. The slowest growing country was Zambia, with an negative rate at −1.8%; Zambia's residents show their income shrinking to half between 1960 and 2000.

- **Figure 1.3** shows an example of how persistent the differences in growth rates were across countries.

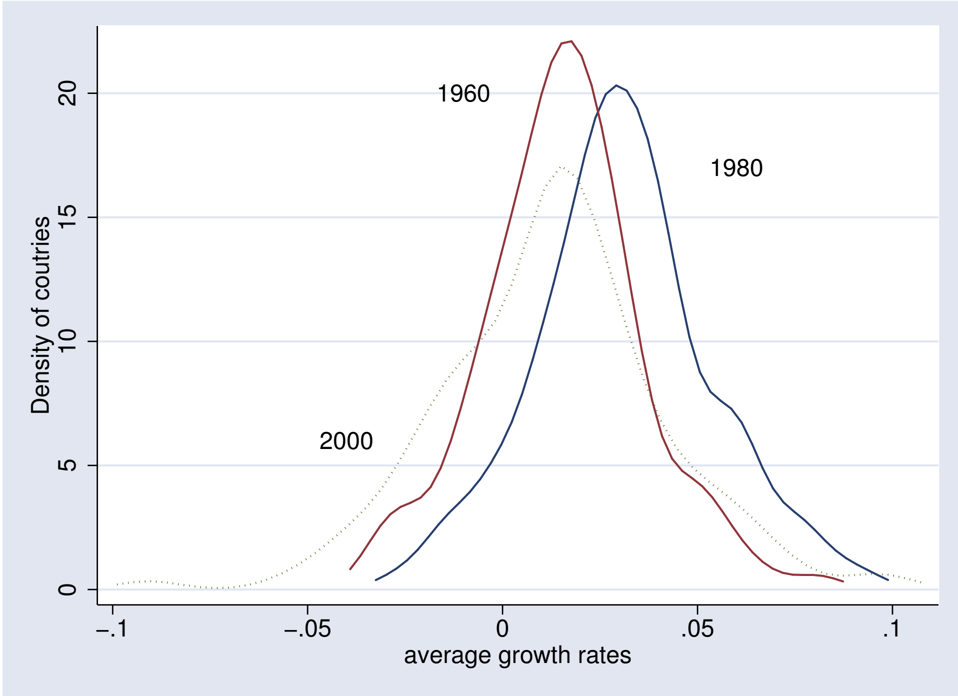

Figure 1.2: Estimates of the distribution of countries according to the growth rate of GDP per worker (PPP-adjusted) in 1960, 1980 and 2000.

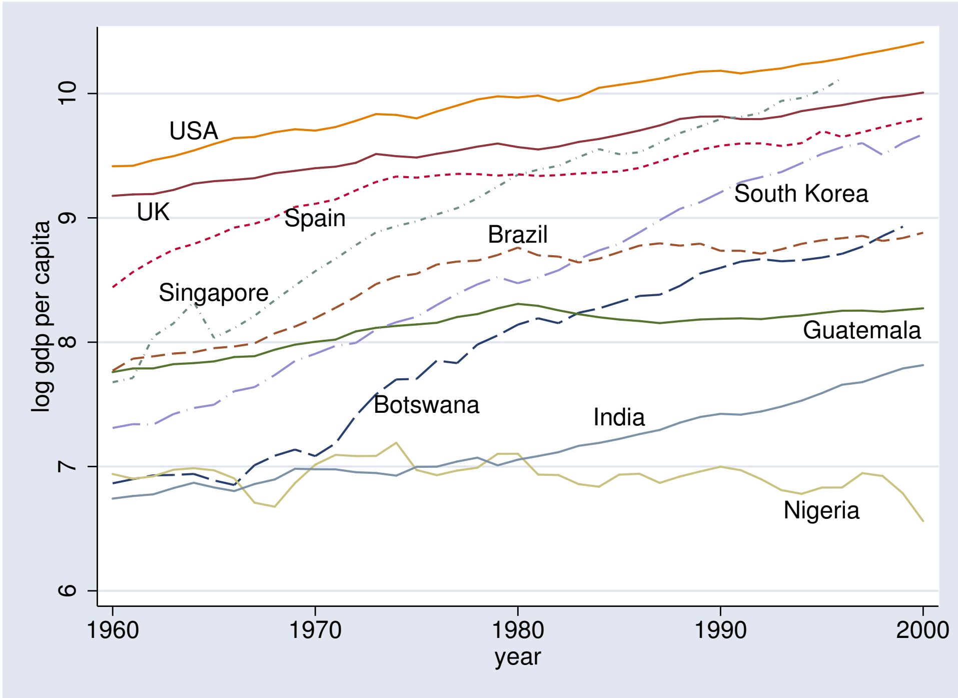

Figure 1.3: The evolution of income per capita in the United States, United Kingdom, Spain, Singapore, Brazil, Guatemala, South Korea, Botswana, Nigeria and India, 1960-2000.

- Most East Asian countries (Taiwan, Singapore, South Korea, Hong Kong, Thailand, China, and Japan), together with Bostwana (an outlier for sub-Saharan Africa), Cyprus, Romania, and Mauritus, had the most stellar growth performances; they were the "growth miracles" of our times. Some OECD countries (Ireland, Portugal, Spain, Greece, Luxemburg and Norway) also made it to the top 20 of the growth-rates chart. On the other hand, 18 out of the bottom 20 were sub-Saharan African countries. Other notable "growth disasters" were Venezuela, Chad and Iraq.

1.3 Unconditional versus Conditional Convergence

- **Figure 1.4** graphs a country's GDP per worker in 1988 (normalized by the US level) against the same country's GDP per worker in 1960. Clearly, most countries did not experienced a dramatic change in their relative position in the world income distribution. Therefore, although there are important movements in the world income distribution, income and productivity differences tend to be very persistent.

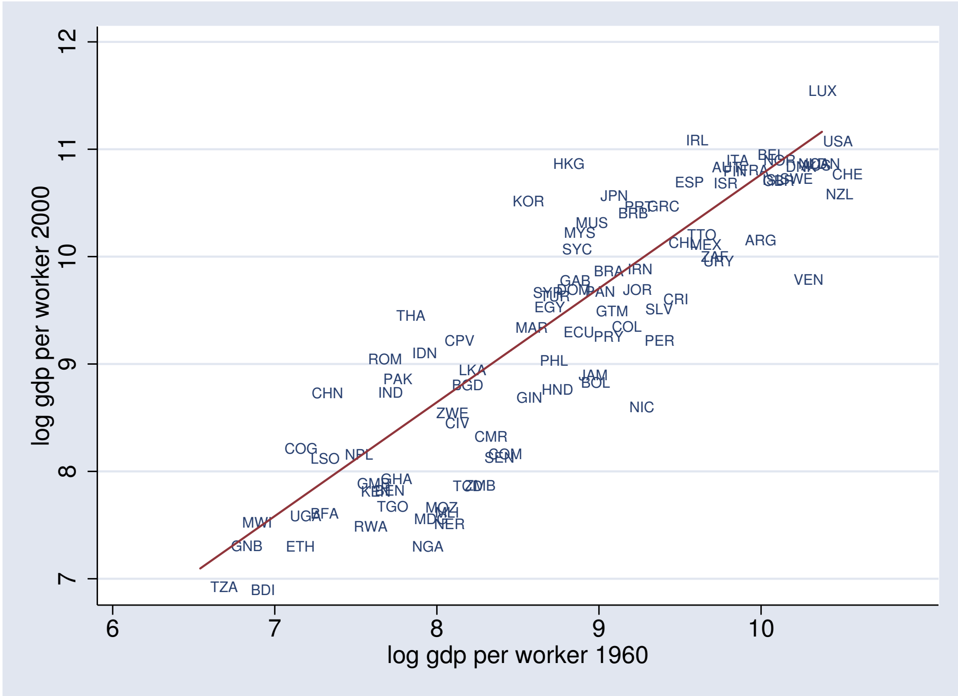

Figure 1.4: Log GDP per worker in 2000 versus log GDP per worker in 1960.

- This also means that *poor countries on average do not grow faster than rich countries.* And another way to state the same fact is that unconditional convergence is zero. That is, if we ran the regression

$$\Delta \ln y_{2000-1960} = \alpha + \beta \cdot \ln y_{1960},$$

the estimated coefficient β is zero.

- On the other hand, consider the regression

$$\Delta \ln y_{1960-90} = \alpha + \beta \cdot \ln y_{1960} + \gamma \cdot X_{1960}$$

where X_{1960} is a set of country-specific controls, such as levels of education, fiscal and monetary policies, market competition, etc. Then, the estimated coefficient β turns to be around 2% per annum. Therefore, poor countries tend to catch up with the rich countries with a group of countries that share similar characteristics. This is what we call *conditional convergence.*

- Conditional convergence is illustrated in **Figure 1.5**, for the group of OECD countries.

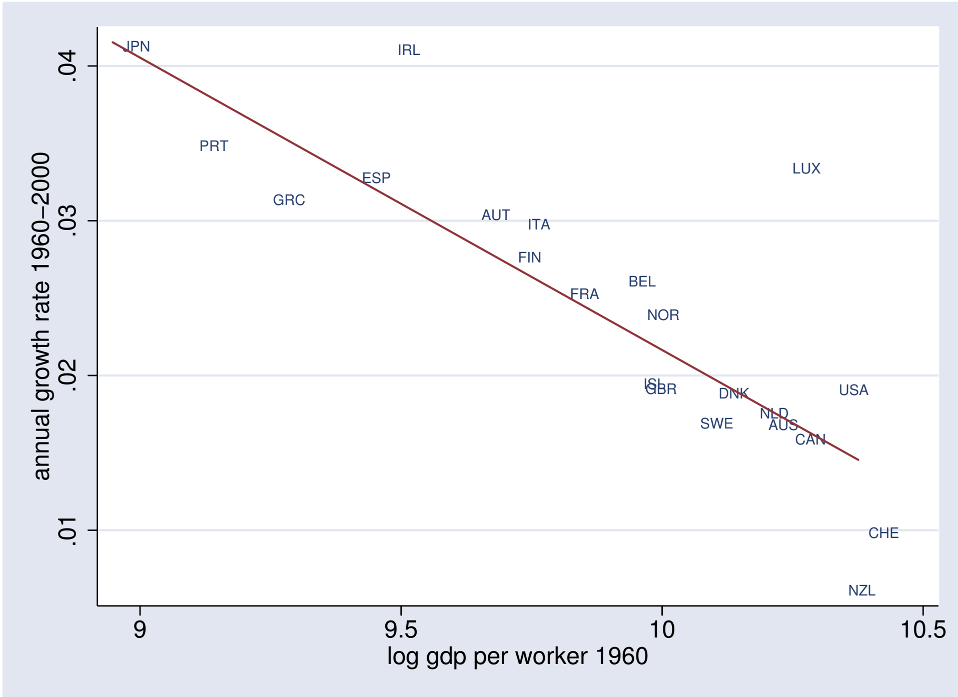

Figure 1.5: Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for core OECD countries.

Chapter 2

The Solow Growth Model

2.1 Centralized Dictatorial Allocations

- In this section, we start the analysis of the Solow model by pretending that there is a dictator, or social planner, that chooses the static and intertemporal allocation of resources and dictates that allocations to the households of the economy We will later show that the allocations that prevail in a decentralized competitive market environment coincide with the allocations dictated by the social planner.

2.1.1 The Economy, the Households and the Dictator

- Time is discrete, $t \in \{0, 1, 2, ...\}$. You can think of the period as a year, as a generation, or as any other arbitrary length of time.

- The economy is an isolated island. Many households live in this island. There are no markets and production is centralized. There is a benevolent dictator, or social planner, who governs all economic and social affairs.

- There is one good, which is produced with two factors of production, capital and labor, and which can be either consumed in the same period, or invested as capital for the next period.

- Households are each endowed with one unit of labor, which they supply inelasticly to the social planner. The social planner uses the entire labor force together with the accumulated aggregate capital stock to produce the one good of the economy.

- In each period, the social planner saves a constant fraction $s \in (0, 1)$ of contemporaneous output, to be added to the economy's capital stock, and distributes the remaining fraction uniformly across the households of the economy.

- In what follows, we let L_t denote the number of households (and the size of the labor force) in period t, K_t aggregate capital stock in the beginning of period t, Y_t aggregate output in period t, C_t aggregate consumption in period t, and I_t aggregate investment in period t. The corresponding lower-case variables represent per-capita measures: $k_t = K_t/L_t$, $y_t = Y_t/L_t$, $i_t = I_t/L_t$, and $c_t = C_t/L_t$.

2.1.2 Technology and Production

- The technology for producing the good is given by

$$Y_t = F(K_t, L_t) \tag{2.1}$$

where $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is a (stationary) production function. We assume that F is continuous and (although not always necessary) twice differentiable.

- We say that the technology is "*neoclassical*" if F satisfies the following properties

1. Constant returns to scale (CRS), or linear homogeneity:

$$F(\mu K, \mu L) = \mu F(K, L), \quad \forall \mu > 0.$$

2. Positive and diminishing marginal products:

$$F_K(K, L) > 0, \quad F_L(K, L) > 0,$$
$$F_{KK}(K, L) < 0, \quad F_{LL}(K, L) < 0.$$

where $F_x \equiv \partial F/\partial x$ and $F_{xz} \equiv \partial^2 F/(\partial x \partial z)$ for $x, z \in \{K, L\}$.

3. Inada conditions:

$$\begin{aligned} \lim_{K \to 0} F_K &= \lim_{L \to 0} F_L = \infty, \\ \lim_{K \to \infty} F_K &= \lim_{L \to \infty} F_L = 0. \end{aligned}$$

- By implication, F satisfies

$$Y = F(K, L) = F_K(K, L)K + F_L(K, L)L$$

or equivalently

$$1 = \varepsilon_K + \varepsilon_L$$

where

$$\varepsilon_K \equiv \frac{\partial F}{\partial K}\frac{K}{F} \quad \text{and} \quad \varepsilon_L \equiv \frac{\partial F}{\partial L}\frac{L}{F}$$

Also, F_K and F_L are homogeneous of degree zero, meaning that the marginal products depend only on the ratio K/L.

And, $F_{KL} > 0$, meaning that capital and labor are complementary.

Finally, all inputs are essential: $F(0, L) = F(K, 0) = 0$.

- Technology in intensive form: Let $y \equiv Y/L$ and $k \equiv K/L$. Then, by CRS

$$y = f(k) \equiv F(k, 1) \tag{2.2}$$

By definition of f and the properties of F, we have that $f(0) = 0$,

$$f'(k) > 0 > f''(k) \qquad \lim_{k \to 0} f'(k) = \infty \qquad \lim_{k \to \infty} f'(k) = 0$$

$$F_K(K, L) = f'(k) \qquad F_L(K, L), = f(k) - f'(k)k$$

- *Example: Cobb-Douglas technology*

$$F(K, L) = K^{\alpha} L^{1-\alpha}$$

In this case, $\varepsilon_K = \alpha$, $\varepsilon_L = 1 - \alpha$, and

$$f(k) = k^{\alpha}$$

2.1.3 The Resource Constraint, and the Law of Motions for Capital and Labor (Population)

- The sum of aggregate consumption and aggregate investment can not exceed aggregate output. That is, the social planner faces the following *resource constraint*:

$$C_t + I_t \leq Y_t \tag{2.3}$$

  Equivalently, in per-capita terms:

$$c_t + i_t \leq y_t \tag{2.4}$$

- We assume that population growth is $n \geq 0$ per period:

$$L_t = (1+n)L_{t-1} = (1+n)^t L_0 \tag{2.5}$$

  We normalize $L_0 = 1$.

- Suppose that existing capital depreciates over time at a fixed rate $\delta \in [0, 1]$. The capital stock in the beginning of next period is given by the non-depreciated part of current-period capital, plus contemporaneous investment. That is, *the law of motion for capital* is

$$K_{t+1} = (1 - \delta)K_t + I_t. \tag{2.6}$$

Equivalently, in per-capita terms:

$$(1 + n)k_{t+1} = (1 - \delta)k_t + i_t$$

We can approximately write the above as

$$k_{t+1} \approx (1 - \delta - n)k_t + i_t \tag{2.7}$$

The sum $\delta + n$ can thus be interpreted as the "effective" depreciation rate of per-capita capital. (Remark: This approximation becomes exact in the continuous-time version of the model.)

2.1.4 The Dynamics of Capital and Consumption

- In most of the growth models that we will examine in this class, the key of the analysis will be to derive a dynamic system that characterizes the evolution of aggregate consumption and capital in the economy; that is, a system of difference equations in C_t and K_t (or c_t and k_t). This system is very simple in the case of the Solow model.

- Combining the law of motion for capital (2.6), the resource constraint (2.3), and the technology (2.1), we derive the difference equation for the capital stock:

$$K_{t+1} - K_t \leq F(K_t, L_t) - \delta K_t - C_t \tag{2.8}$$

That is, the change in the capital stock is given by aggregate output, minus capital depreciation, minus aggregate consumption.

$$k_{t+1} - k_t \leq f(k_t) - (\delta + n)k_t - c_t.$$

2.1.5 Feasible and "Optimal" Allocations

Definition 1 *A feasible allocation is any sequence* $\{c_t, k_t\}_{t=0}^{\infty} \in \left(\mathbb{R}_+^2\right)^{\infty}$ *that satisfies the resource constraint*

$$k_{t+1} \leq f(k_t) + (1 - \delta - n)k_t - c_t. \tag{2.9}$$

- The set of feasible allocations represents the "choice set" for the social planner. The planner then uses some choice rule to select one of the many feasible allocations.

- Later, we will have to social planner choose an allocation so as to maximize welfare (Pareto efficiency). Here, we instead assume that the dictator follows a simple rule-of-thump.

Definition 2 *A "Solow-optimal" centralized allocation is any feasible allocation that satisfies the resource constraint with equality and*

$$c_t = (1 - s)f(k_t), \tag{2.10}$$

for some $s \in (0, 1)$.

2.1.6 The Policy Rule

- Combining (2.9) and (2.10) gives a single difference equation that completely characterizes the dynamics of the Solow model.

Proposition 3 *Given any initial point* $k_0 > 0$*, the dynamics of the dictatorial economy are given by the path* $\{k_t\}_{t=0}^{\infty}$ *such that*

$$k_{t+1} = G(k_t), \tag{2.11}$$

for all $t \geq 0$*, where*

$$G(k) \equiv sf(k) + (1 - \delta - n)k.$$

Equivalently, the growth rate is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = \gamma(k_t), \tag{2.12}$$

where

$$\gamma(k) \equiv s\phi(k) - (\delta + n), \quad \phi(k) \equiv f(k)/k.$$

- G corresponds to what we will call the *policy rule* in the Ramsey model. The dynamic evolution of the economy is concisely represented by the path $\{k_t\}_{t=0}^{\infty}$ that satisfies (**??**), or equivalently (2.11), for all $t \geq 0$, with k_0 historically given.

- The graph of G is illustrated in **Figure 2.1**.

- *Remark.* Think of G more generally as a function that tells you what is the state of the economy tomorrow as a function of the state today. Here and in the simple Ramsey model, the state is simply k_t. When we introduce productivity shocks, the state is (k_t, A_t). When we introduce multiple types of capital, the state is the vector of capital stocks. And with incomplete markets, the state is the whole distribution of wealth in the cross-section of agents.

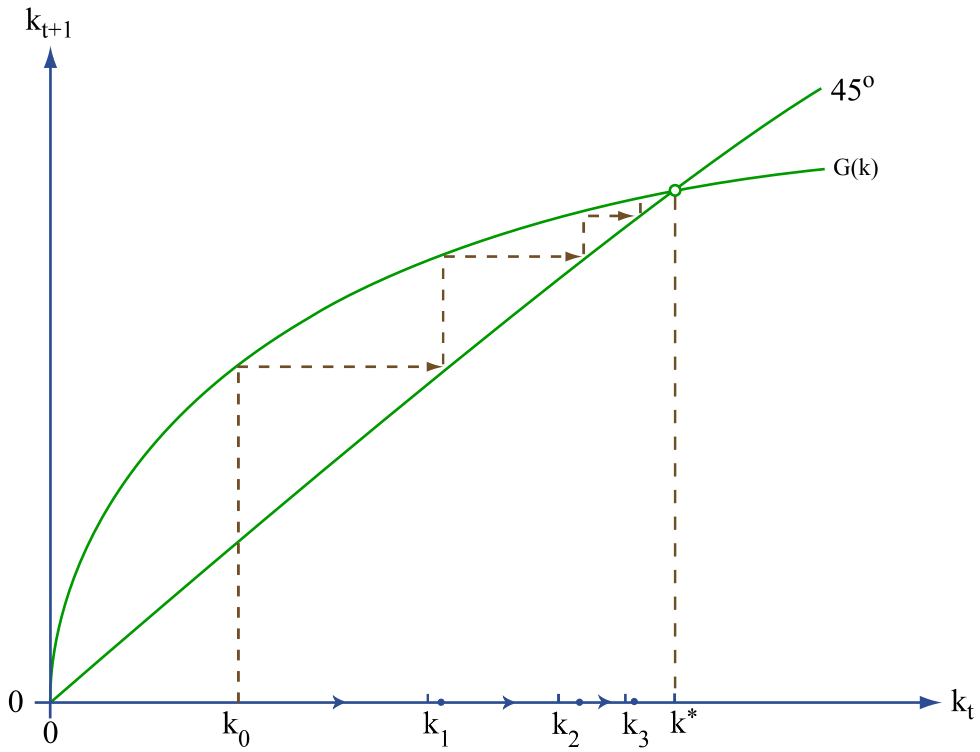

Figure 2.1: The policy rule $k_{t+1} = G(k_t)$ of the Solow model.

Figure by MIT OCW.

2.1.7 Steady State

- A *steady state* of the economy is defined as any level k^* such that, if the economy starts with $k_0 = k^*$, then $k_t = k^*$ for all $t \geq 1$. That is, a steady state is any fixed point k^* of G in (2.11). Equivalently, a steady state is any fixed point (c^*, k^*) of the system (2.9)-(2.10).

- A trivial steady state is $c = k = 0$: There is no capital, no output, and no consumption. This would not be a steady state if $f(0) > 0$. We are interested for steady states at which capital, output and consumption are all positive and finite. We can easily show:

Proposition 4 *Suppose $\delta + n \in (0, 1)$ and $s \in (0, 1)$. A steady state $(c^*, k^*) \in (0, \infty)^2$ for the dictatorial economy exists and is unique. k^* and y^* increase with s and decrease with δ and n, whereas c^* is non-monotonic with s and decreases with δ and n. Finally, $y^*/k^* = (\delta + n)/s$.*

- *Proof.* k^* is a steady state if and only if it solves

$$0 = sf(k^*) - (\delta + n)k^*,$$

Equivalently

$$\phi(k^*) = \frac{\delta + n}{s} \tag{2.13}$$

where $\phi(k) \equiv \frac{f(k)}{k}$. The function ϕ gives the output-to-capital ratio in the economy. The properties of f imply that ϕ is continuous and strictly decreasing, with

$$\phi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} < 0,$$

$$\phi(0) = f'(0) = \infty \quad \text{and} \quad \phi(\infty) = f'(\infty) = 0,$$

where the latter follow from L'Hospital's rule. This implies that equation (2.13) has a unique solution:

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right).$$

Since $\phi' < 0$, k^* is a decreasing function of $(\delta + n)/s$. On the other hand, consumption is given by$c^* = (1 - s)f(k^*)$. It follows that c^* decreases with $\delta + n$, but s has an ambiguous effect.

2.1.8 Transitional Dynamics

- The above characterized the (unique) steady state of the economy. Naturally, we are interested to know whether the economy will converge to the steady state if it starts away from it. Another way to ask the same question is whether the economy will eventually return to the steady state after an exogenous shock perturbs the economy and moves away from the steady state.

- The following uses the properties of G to establish that, in the Solow model, convergence to the steady is always ensured and is monotonic:

Proposition 5 *Given any initial* $k_0 \in (0, \infty)$, *the dictatorial economy converges asymptotically to the steady state. The transition is monotonic. The growth rate is positive and decreases over time towards zero if* $k_0 < k^*$; *it is negative and increases over time towards zero if* $k_0 > k^*$.

- *Proof.* From the properties of f, $G'(k) = sf'(k) + (1 - \delta - n) > 0$ and $G''(k) = sf''(k) < 0$. That is, G is strictly increasing and strictly concave. Moreover, $G(0) = 0$ and $G(k^*) = k^*$. It follows that $G(k) > k$ for all $k < k^*$ and $G(k) < k$ for all $k > k^*$. It follows that $k_t < k_{t+1} < k^*$ whenever $k_t \in (0, k^*)$ and therefore the sequence $\{k_t\}_{t=0}^{\infty}$ is strictly increasing if $k_0 < k^*$. By monotonicity, k_t converges asymptotically to some $\hat{k} \leq k^*$. By continuity of G, $\hat{k}$ must satisfy $\hat{k} = G(\hat{k})$, that is $\hat{k}$ must be a fixed point of G. But we already proved that G has a unique fixed point, which proves that $\hat{k} = k^*$. A symmetric argument proves that, when $k_0 > k^*$, $\{k_t\}_{t=0}^{\infty}$ is stricttly decreasing and again converges asymptotically to k^*. Next, consider the growth rate of the capital stock. This is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = s\phi(k_t) - (\delta + n) \equiv \gamma(k_t).$$

  Note that $\gamma(k) = 0$ iff $k = k^*$, $\gamma(k) > 0$ iff $k < k^*$, and $\gamma(k) < 0$ iff $k > k^*$. Moreover, by diminishing returns, $\gamma'(k) = s\phi'(k) < 0$. It follows that $\gamma(k_t) < \gamma(k_{t+1}) < \gamma(k^*) = 0$ whenever $k_t \in (0, k^*)$ and $\gamma(k_t) > \gamma(k_{t+1}) > \gamma(k^*) = 0$ whenever $k_t \in (k^*, \infty)$. This proves that γ_t is positive and decreases towards zero if $k_0 < k^*$ and it is negative and increases towards zero if $k_0 > k^*$. ■

- **Figure 2.1** depicts $G(k)$, the relation between k_t and k_{t+1}. The intersection of the graph of G with the 45^o line gives the steady-state capital stock k^*. The arrows represent the path $\{k_t\}_{t=}^{\infty}$ for a particular initial k_0.

- **Figure 2.2** depicts $\gamma(k)$, the relation between k_t and γ_t. The intersection of the graph of γ with the 45^o line gives the steady-state capital stock k^*. The negative slope reflects what we call "conditional convergence."

- Discuss local versus global stability: Because $\phi'(k^*) < 0$, the system is locally stable. Because ϕ is globally decreasing, the system is globally stable and transition is monotonic.

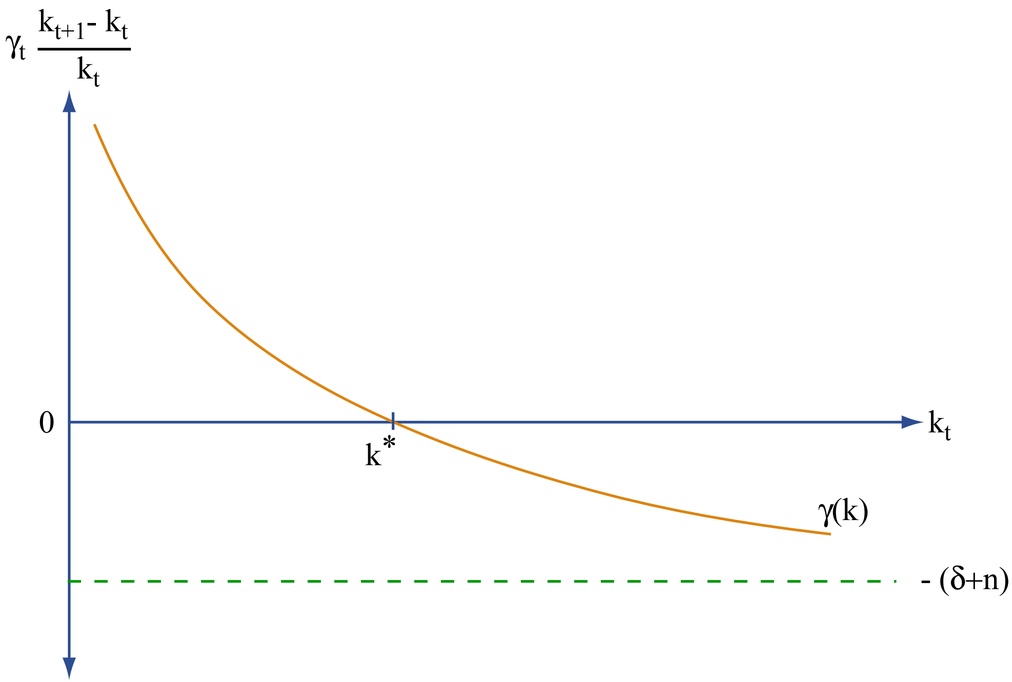

Figure 2.2. The growth rate $\frac{k_{t+1} - k_t}{k_t} = \gamma(k_t)$ in the Solow model.

Figure by MIT OCW.

2.2 Decentralized Market Allocations

- In the previous section, we characterized the centralized allocation dictated by a social planner. We now characterize the competitive market allocation

2.2.1 Households

- Households are dynasties, living an infinite amount of time. We index households by $j \in [0, 1]$, having normalized $L_0 = 1$.

- The number of heads in every household grow at constant rate $n \geq 0$. Therefore, the size of the population in period t is $L_t = (1 + n)^t$ and the number of persons in each household in period t is also L_t.

- We write $c_t^j, k_t^j, b_t^j, i_t^j$ for the per-head variables for household j.

- Each person in a household is endowed with one unit of labor in every period, which he supplies inelastically in a competitive labor market for the contemporaneous wage w_t. Household j is also endowed with initial capital k_0^j. Capital in household j accumulates according to

$$(1+n)k_{t+1}^j = (1-\delta)k_t^j + i_t,$$

which we approximate by

$$k_{t+1}^j = (1-\delta-n)k_t^j + i_t. \tag{2.14}$$

Households rent the capital they own to firms in a competitive market for a (gross) rental rate r_t.

- The household may also hold stocks of some firms in the economy. Let π_t^j be the dividends (firm profits) that household j receive in period t. It is without any loss of generality to assume that there is no trade of stocks (because the value of stocks will be zero in equilibrium). We thus assume that household j holds a fixed fraction α^j of the aggregate index of stocks in the economy, so that $\pi_t^j = \alpha^j \Pi_t$, where Π_t are aggregate profits. Of course, $\int \alpha^j dj = 1$.

- The household uses its income to finance either consumption or investment in new capital:

$$c_t^j + i_t^j = y_t^j.$$

Total per-head income for household j in period t is simply

$$y_t^j = w_t + r_t k_t^j + \pi_t^j. \tag{2.15}$$

Combining, we can write the budget constraint of household j in period t as

$$c_t^j + i_t^j = w_t + r_t k_t^j + \pi_t^j \tag{2.16}$$

- Finally, the consumption and investment behavior of household is a simplistic linear rule. They save fraction s and consume the rest:

$$c_t^j = (1-s)y_t^j \qquad \text{and} \qquad i_t^j = sy_t^i. \tag{2.17}$$

2.2.2 Firms

- There is an arbitrary number M_t of firms in period t, indexed by $m \in [0, M_t]$. Firms employ labor and rent capital in competitive labor and capital markets, have access to the same neoclassical technology, and produce a homogeneous good that they sell competitively to the households in the economy.

- Let K_t^m and L_t^m denote the amount of capital and labor that firm m employs in period t. Then, the profits of that firm in period t are given by

$$\Pi_t^m = F(K_t^m, L_t^m) - r_t K_t^m - w_t L_t^m.$$

- The firms seek to maximize profits. The FOCs for an interior solution require

$$F_K(K_t^m, L_t^m) = r_t. \tag{2.18}$$

$$F_L(K_t^m, L_t^m) = w_t. \tag{2.19}$$

- Remember that the marginal products are homogenous of degree zero; that is, they depend only on the capital-labor ratio. In particular, F_K is a decreasing function of K_t^m/L_t^m and F_L is an increasing function of K_t^m/L_t^m. Each of the above conditions thus pins down a unique capital-labor ratio K_t^m/L_t^m. For an interior solution to the firms' problem to exist, it must be that r_t and w_t are consistent, that is, they imply the same K_t^m/L_t^m. This is the case if and only if there is some $X_t \in (0, \infty)$ such that

$$r_t = f'(X_t) \tag{2.20}$$

$$w_t = f(X_t) - f'(X_t)X_t \tag{2.21}$$

  where $f(k) \equiv F(k,1)$; this follows from the properties $F_K(K,L) = f'(K/L)$ and $F_L(K,L) = f(K/L) - f'(K/L) \cdot (K/L)$, which we established earlier. That is, (w_t, r_t) must satisfy $w_t = W(r_t)$ where $W(r) \equiv f\left(f'^{-1}(r)\right) - rf'^{-1}(r)$.

- If (2.20) and (2.21) are satisfied, the FOCs reduce to $K_t^m/L_t^m = X_t$, or

$$K_t^m = X_t L_t^m. \tag{2.22}$$

That is, the FOCs pin down the capital-labor ratio for each firm (K_t^m/L_t^m), but not the size of the firm (L_t^m). Moreover, all firms use the same capital-labor ratio.

- Besides, (2.20) and (2.21) imply

$$r_t X_t + w_t = f(X_t). \tag{2.23}$$

It follows that

$$r_t K_t^m + w_t L_t^m = (r_t X_t + w_t) L_t^m = f(X_t) L_t^m = F(K_t^m, L_t^m),$$

and therefore

$$\Pi_t^m = L_t^m [f(X_t) - r_t X_t - w_t] = 0. \tag{2.24}$$

That is, when (2.20) and (2.21) are satisfied, the maximal profits that any firm makes are exactly zero, and these profits are attained for any firm size as long as the capital-labor ratio is optimal. If instead (2.20) and (2.21) were violated, then either $r_t X_t + w_t < f(X_t)$, in which case the firm could make infinite profits, or $r_t X_t + w_t > f(X_t)$, in which case operating a firm of any positive size would generate strictly negative profits.

2.2.3 Market Clearing

- The *capital market* clears if and only if

$$\int_0^{M_t} K_t^m dm = \int_0^1 (1+n)^t k_t^j dj$$

Equivalently,

$$\int_0^{M_t} K_t^m dm = K_t \tag{2.25}$$

where $K_t \equiv \int_0^{L_t} k_t^j dj$ is the aggregate capital stock in the economy.

- The *labor market*, on the other hand, clears if and only if

$$\int_0^{M_t} L_t^m dm = \int_0^1 (1+n)^t dj$$

Equivalently,

$$\int_0^{M_t} L_t^m dm = L_t \tag{2.26}$$

2.2.4 General Equilibrium: Definition

- The definition of a *general equilibrium* is more meaningful when households optimize their behavior (maximize utility) rather than being automata (mechanically save a constant fraction of income). Nonetheless, it is always important to have clear in mind what is the definition of equilibrium in any model. For the decentralized version of the Solow model, we let:

Definition 6 *An equilibrium of the economy is an allocation* $\{(k_t^j, c_t^j, i_t^j)_{j\in[0,1]}, (K_t^m, L_t^m)_{m\in[0,M_t]}\}_{t=0}^{\infty}$*, a distribution of profits* $\{(\pi_t^j)_{j\in[0,1]}\}$*, and a price path* $\{r_t, w_t\}_{t=0}^{\infty}$ *such that*

(i) Given $\{r_t, w_t\}_{t=0}^{\infty}$ *and* $\{\pi_t^j\}_{t=0}^{\infty}$*, the path* $\{k_t^j, c_t^j, i_t^j\}$ *is consistent with the behavior of household* j*, for every* j*.*

(ii) (K_t^m, L_t^m) *maximizes firm profits, for every* m *and* t*.*

(iii) The capital and labor markets clear in every period

2.2.5 General Equilibrium: Characterization

Proposition 7 *For any initial positions* $(k_0^j)_{j\in[0,1]}$, *an equilibrium exists. The allocation of production across firms is indeterminate, but the equilibrium is unique with regard to aggregates and household allocations. The capital-labor ratio in the economy is given by* $\{k_t\}_{t=0}^{\infty}$ *such that, for all* $t \geq 0$,

$$k_{t+1} = G(k_t) \tag{2.27}$$

with $k_0 = \int k_0^j dj$ *given and with* $G(k) \equiv sf(k) + (1-\delta-n)k$. *Equilibrium growth is*

$$\gamma_t \equiv \frac{k_{t+1}-k_t}{k_t} = \gamma(k_t), \tag{2.28}$$

where $\gamma(k) \equiv s\phi(k) - (\delta+n)$, $\phi(k) \equiv f(k)/k$. *Finally, equilibrium prices are given by*

$$r_t = r(k_t) \equiv f'(k_t), \tag{2.29}$$

$$w_t = w(k_t) \equiv f(k_t) - f'(k_t)k_t, \tag{2.30}$$

- *Proof.* We first characterize the equilibrium, assuming it exists. Using $K_t^m = X_t L_t^m$ by (2.22), we can write the aggregate demand for capital as $\int_0^{M_t} K_t^m dm = X_t \int_0^{M_t} L_t^m dm$. From the labor market clearing condition (2.26), $\int_0^{M_t} L_t^m dm = L_t$. Combining, we infer $\int_0^{M_t} K_t^m dm = X_t L_t$, and substituting in the capital market clearing condition (2.25), we conclude $X_t L_t = K_t$, where $K_t \equiv \int_0^{L_t} k_t^j dj$ denotes the aggregate capital stock. Equivalently, letting $k_t \equiv K_t / L_t$ denote the capital-labor ratio in the economy, we have

$$X_t = k_t. \tag{2.31}$$

That is, all firms use the same capital-labor ratio as the aggregate of the economy.

Substituting (2.31) into (2.20) and (2.21) we infer that equilibrium prices are given by

$$\begin{aligned} r_t &= r(k_t) \equiv f'(k_t) = F_K(k_t, 1) \\ w_t &= w(k_t) \equiv f(k_t) - f'(k_t)k_t = F_L(k_t, 1) \end{aligned}$$

Note that $r'(k) = f''(k) = F_{KK} < 0$ and $w'(k) = -f''(k)k = F_{LK} > 0$. That is, the interest rate is a decreasing function of the capital-labor ratio and the wage rate is an increasing function of the capital-labor ratio. The first properties reflects diminishing returns, the second reflects the complementarity of capital and labor.

Adding up the budget constraints of the households, we get

$$C_t + I_t = r_t K_t + w_t L_t + \int \pi_t^j dj,$$

where $C_t \equiv \int c_t^j dj$ and $I_t \equiv \int i_t^j dj$. Aggregate dividends must equal aggregate profits, $\int \pi_t^j dj = \int \Pi_t^m dj$. By (2.24), profits for each firm are zero. Therefore, $\int \pi_t^j dj = 0$, implying $C_t + I_t = Y_t = r_t K_t + w_t L_t$. Equivalently, in per-capita terms,

$$c_t + i_t = y_t = r_t k_t + w_t.$$

From (2.23) and (2.31), or equivalently from (2.29) and (2.30),

$$r_t k_t + w_t = y_t = f(k_t)$$

We conclude that the household budgets imply

$$c_t + i_t = f(k_t),$$

which is simply the resource constraint of the economy.

Adding up the individual capital accumulation rules (2.14), we get the capital accumulation rule for the aggregate of the economy. In per-capita terms,

$$k_{t+1} = (1 - \delta - n)k_t + i_t$$

Adding up (2.17) across households, we similarly infer $i_t = sy_t = sf(k_t)$. Combining, we conclude

$$k_{t+1} = sf(k_t) + (1 - \delta - n)k_t = G(k_t),$$

which is exactly the same as in the centralized allocation.

Finally, existence and uniqueness is now trivial. (2.27) maps any $k_t \in (0,\infty)$ to a unique $k_{t+1} \in (0,\infty)$. Similarly, (2.29) and (2.30) map any $k_t \in (0,\infty)$ to unique $r_t, w_t \in (0,\infty)$. Therefore, given any initial $k_0 = \int k_0^j dj$, there exist unique paths $\{k_t\}_{t=0}^\infty$ and $\{r_t, w_t\}_{t=0}^\infty$. Given $\{r_t, w_t\}_{t=0}^\infty$, the allocation $\{k_t^j, c_t^j, i_t^j\}$ for any household j is then uniquely determined by (2.14), (2.15), and (2.17). Finally, any allocation $(K_t^m, L_t^m)_{m\in[0,M_t]}$ of production across firms in period t is consistent with equilibrium as long as $K_t^m = k_t L_t^m$. ■

Corollary 8 *The aggregate and per-capita allocations in the competitive market economy coincide with those in the dictatorial economy.*

- We can thus immediately translate the steady state and the transitional dynamics of the centralized plan to the steady state and the transitional dynamics of the decentralized market allocations.

- Remark: This example is just a prelude to the first and second welfare theorems, which we will have once we replace the "rule-of-thumb" behavior of the households with optimizing behavior given a preference ordering over different consumption paths: in the neoclassical growth model, Pareto efficient and competitive equilibrium allocations coincide.

2.3 Shocks and Policies

- The Solow model can be interpreted also as a primitive Real Business Cycle (RBC) model. We can use the model to predict the response of the economy to productivity, taste, or policy shocks.

2.3.1 Productivity (or Taste) Shocks

- Suppose output is given by

$$Y_t = A_t F(K_t, L_t)$$

  or $y_t = A_t f(k_t)$, where A_t denotes total factor productivity.

- Consider a permanent negative shock in A. The $G(k)$ and $\gamma(k)$ functions shift down, as illustrated in Figure 2.3. The economy transits slowly from the old steady state to the new, lower steady state.

- If instead the shock is transitory, the shift in $G(k)$ and $\gamma(k)$ is also temporary. Initially, capital and output fall towards the low steady state. But when productivity reverts to the initial level, capital and output start to grow back towards the old high steady state.

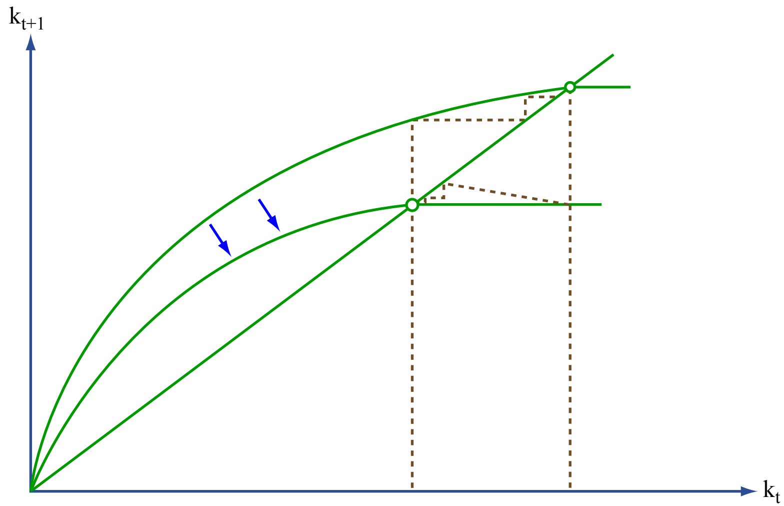

Figure 2.3: The impact of a productivity shock on G.

Figure by MIT OCW.

- The effect of a prodictivity shock on k_t and y_t is illustrated in Figure 2.4. The solid lines correspond to a transitory shock, whereas the dashed lines correspond to a permanent shock.

- *Taste shocks:* Consider a temporary fall in the saving rate s. The $\gamma(k)$ function shifts down for a while, and then return to its initial position. What are the transitional dynamics? What if instead the fall in s is permanent?

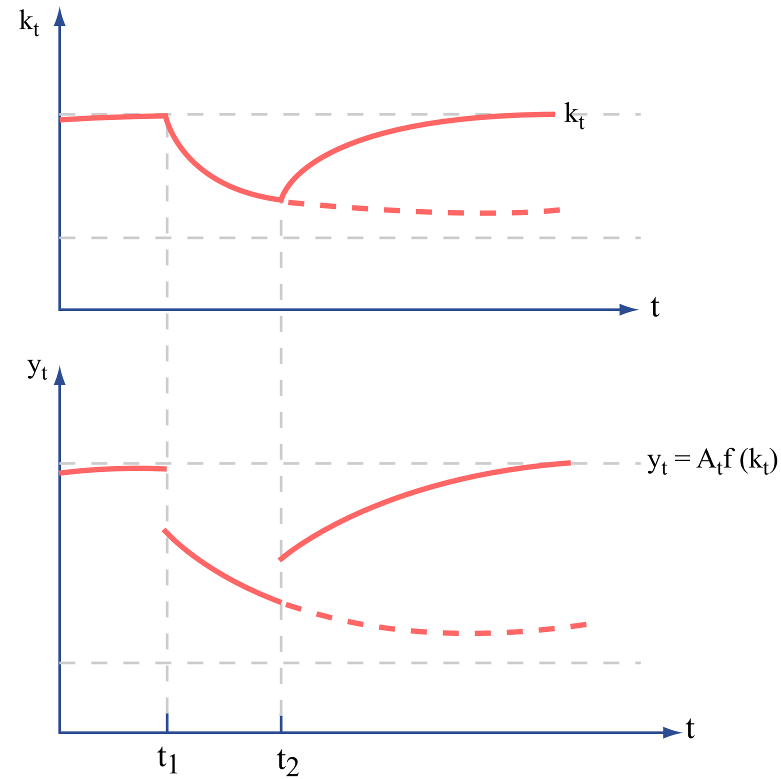

Figure 2.4: The impact of a productivity shock on G.

Figure by MIT OCW.

2.3.2 Unproductive Government Spending

- Let us now introduce a *government* in the competitive market economy. The government spends resources without contributing to production or capital accumulation. The resource constraint of the economy now becomes

$$c_t + g_t + i_t = y_t = f(k_t),$$

where g_t denotes government consumption. The latter is financed with proportional income taxation: $g_t = \tau y_t$.

- Disposable income for the representative household is $(1-\tau)y_t$. We continue to assume agents consume a fraction s of disposable income: $i_t = (1-s)(y_t - g_t)$.

- Combining the above, we conclude that the dynamics of capital are now given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1-\tau)\phi(k_t) - (\delta + n).$$

where $\phi(k) \equiv f(k)/k$. Given s and k_t, the growth rate γ_t decreases with τ.

- A steady state exists for any $\tau \in [0, 1)$ and is given by

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s(1-\tau)}\right).$$

Given s, k^* decreases with τ.

- *Policy Shocks:* Consider a temporary shock in government consumption. What are the transitional dynamics?

2.3.3 Productive Government Spending

- Suppose now that production is given by

$$y_t = f(k_t, g_t) = k_t^{\alpha} g_t^{\beta},$$

where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$. Government spending can thus be interpreted as infrastructure or other productive services. The resource constraint is $c_t + g_t + i_t = y_t = f(k_t, g_t)$.

- Government spending is financed with proportional income taxation and private consumption is a fraction $1-s$ of disposable income: $g_t = \tau y_t$, $c_t = (1-s)(y_t - g_t)$, $i_t = s(y_t - g_t)$.

- Substituting $g_t = \tau y_t$ into $y_t = k_t^\alpha g_t^\beta$ and solving for y_t, we infer

$$y_t = k_t^{\frac{\alpha}{1-\beta}} \tau^{\frac{\beta}{1-\beta}} \equiv k_t^a \tau^b$$

where $a \equiv \alpha/(1-\beta)$ and $b \equiv \beta/(1-\beta)$. Note that $a > \alpha$.

- We conclude that the dynamics and the steady state are given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1-\tau)\tau^b k_t^{a-1} - (\delta + n).$$

$$k^* = \left(\frac{s(1-\tau)\tau^b}{\delta + n} \right)^{1/(1-a)}.$$

- Consider the rate τ that maximizes either k^*, or γ_t for any given k_t. This is given by $\tau = b/(1+b) = \beta$. The more productive government services are, the higher their "optimal" provision.

2.4 Continuous Time, Long-Run Growth, Convergence

2.4.1 The Solow model in continuous time

- We now consider the model in continuous time and also allow for exogenous growth.

- The resource constraint of the economy is

$$C + I = Y = F(K, AL).$$

Population growth and exogenous technological change are given by

$$\frac{\dot{L}}{L} = n \quad \text{and} \quad \frac{\dot{A}}{A} = x$$

and the law of motion for aggregate capital is

$$\dot{K} = I - \delta K$$

- Let $k \equiv K/(AL)$, $y = Y/(AL)$, etc. Then, $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} - \frac{\dot{A}}{A}$. Substituting from the above, we infer

$$\dot{k} = i - (\delta + n + x)k.$$

Combining this with $i = sy = sf(k)$, we conclude

$$\dot{k} = sf(k) - (\delta + n + n)k.$$

- Equivalently, the growth rate of the economy is given by

$$\frac{\dot{k}}{k} = \gamma(k) \equiv s\phi(k) - (\delta + n + x). \tag{2.32}$$

This is the same $\gamma(k)$ function as in the discrete-time version.

- Exogenous productivity growth x simply contributes to the effective depreciation of k, the capital-to-effective-labor ratio.

2.4.2 Log-linearization and the Convergence Rate

- Define $z \equiv \ln k - \ln k^*$. We can rewrite the growth equation (2.32) as

$$\dot{z} = \Gamma(z),$$
$$\Gamma(z) \equiv \gamma(k^* e^z) \equiv s\phi(k^* e^z) - (\delta + n + x)$$

  Note that $\Gamma(z)$ is defined for all $z \in \mathbb{R}$. By definition of k^*, $\Gamma(0) = s\phi(k^*) - (\delta + n + x) = 0$. Similarly, $\Gamma(z) > 0$ for all $z < 0$ and $\Gamma(z) < 0$ for all $z > 0$. Finally, $\Gamma'(z) = s\phi'(k^* e^z)k^* e^z < 0$ for all $z \in \mathbb{R}$.

- We next linearize $\dot{z} = \Gamma(z)$ around $z = 0$:

$$\dot{z} = \Gamma(0) + \Gamma'(0) \cdot z$$
$$\dot{z} = \lambda z$$

  where we substituted $\Gamma(0) = 0$ and let $\lambda \equiv \Gamma'(0)$.

- Straightforward algebra gives

$$
\begin{aligned}
\Gamma'(z) &= s\phi'(k^* e^z)k^* e^z < 0 \\
\phi'(k) &= \frac{f'(k)k - f(k)}{k^2} = -\left[1 - \frac{f'(k)k}{f(k)}\right]\frac{f(k)}{k^2} \\
sf(k^*) &= (\delta + n + x)k^* \\
\Gamma'(0) &= -(1 - \varepsilon_K)(\delta + n + x) < 0
\end{aligned}
$$

  where $\varepsilon_K \equiv F_K K/F = f'(k)k/f(k)$ is the elasticity of production with respect to capital, evaluated at the steady-state k.

- We conclude that

$$
\frac{\dot{k}}{k} = \lambda \ln\left(\frac{k}{k^*}\right)
$$

  where

$$
\lambda = -(1 - \varepsilon_K)(\delta + n + x) < 0
$$

  The quantity $-\lambda$ is called the *convergence rate.*

- In the Cobb-Douglas case, $y = k^{\alpha}$, the convergence rate is simply

$$-\lambda = (1-\alpha)(\delta + n + x),$$

- Note that as $\lambda \to 0$ as $\alpha \to 1$. That is, convergence becomes slower and slower as the income share of capital becomes closer and closer to 1. Indeed, if it were $\alpha = 1$, the economy would a balanced growth path.

- Around the steady state $\frac{\dot{y}}{y} = \varepsilon_K \cdot \frac{\dot{k}}{k}$ and $\frac{y}{y^*} = \varepsilon_K \cdot \frac{k}{k^*}$ It follows that

$$\frac{\dot{y}}{y} = \lambda \ln\left(\frac{y}{y^*}\right)$$

  Thus, $-\lambda$ is the convergence rate for either capital or output.

- *Quantitative:* If $\alpha = 35\%$, $n = 1\%$, $x = 2\%$, and $\delta = 5\%$, then $-\lambda = 6\%$. This clearly contradicts the data. But if $\alpha = 70\%$, then $-\lambda = 2.4\%$, which does a better job in matching the data on conditional convergence. (Hint: think of a broad definition of capital.)

2.5 Cross-Country Differences: Mankiw-Romer Weil

- The Solow model implies that steady-state capital, productivity, and income are determined by A, δ, x and s, n. Assuming that countries share the same technology up to a level difference, if the Solow model is correct, observed cross-country income and productivity differences should be "explained" by observed cross-country differences in s and n.

- Mankiw, Romer and Weil (1992) tests this hypothesis against the data. In its simple form, the Solow model fails to predict the large cross-country dispersion of income and productivity levels. They then consider an extension of the Solow model, that includes two types of capital, physical capital (K) and human capital (H). Output is given by

$$Y_t = f\left(K_t, H_t, A_t L_t\right) = K_t^{\alpha} H_t^{\beta} \left(A_t L_t\right)^{1-\beta},$$

where $\alpha > 0, \beta > 0$, and $\alpha + \beta < 1$. Equivalently,

$$y_t = f\left(k_t, h_t\right) = k_t^{\alpha} h_t^{\beta}.$$

- The dynamics of capital accumulation are now given by

$$\begin{aligned} \dot{k}_t &= s_k f\left(k_t, h_t\right) - (\delta_k + n + x)k_t \\ \dot{h}_t &= s_h f\left(k_t, h_t\right) - (\delta_h + n + x)h_t \end{aligned}$$

where s_k and s_h are the investment rates in physical capital and human capital, respectively, and δ_k and δ_h are the respective depreciation rates. We henceforth assume $\delta_k = \delta_h = \delta$.

- The steady-state levels of k, h, and y then depend on both s_k and s_h, as well as δ and n. Letting j index a country, and assuming that δ and x are the same across countries, while s_k, s_h and n differ, we have

$$\begin{aligned} k_j^* &= \left(\left(\frac{s_{k,j}}{n_j + x + \delta}\right)^{1-\beta}\left(\frac{s_{h,j}}{n_j + x + \delta}\right)^{\beta}\right)^{\frac{1}{1-\alpha-\beta}} \\ h_j^* &= \left(\left(\frac{s_{k,j}}{n_j + x + \delta}\right)^{\alpha}\left(\frac{s_{h,j}}{n_j + x + \delta}\right)^{1-\alpha}\right)^{\frac{1}{1-\alpha-\beta}}. \end{aligned}$$

- Along the steady state,

$$\left(\frac{Y}{L}\right)_{j,t} = A_{j,t} f\left(k_j^*, h_j^*\right).$$

Letting $A_{j,t} = \exp\left(xt + a_j\right)$, where a_j is the level of technology for country j, we get

$$\ln\left(\frac{Y}{L}\right)_{j,t} = a_j + xt + b_1 \ln s_{kj} + b_2 \ln s_{hj} + b_3 \ln\left(n_j + x + \delta\right) \tag{2.33}$$

$$b_1 = \frac{\alpha}{1-\alpha-\beta}, \quad b_2 = \frac{\beta}{1-\alpha-\beta}, \quad b_3 = -\frac{\alpha+\beta}{1-\alpha-\beta}.$$

- The idea is then to run a regression like (2.33) in cross-country data. To do this, one needs (i) to specify a measure for s_{hj}, (ii) to assume that x is common across countries, and (iii) to assume that a_j is orthogonal to $\left(s_{kl}, s_{hj}, n_j\right)$. The last assumption is particularly problematic. As for what measure of s_h to use, that's another tricky point. But let's ignore these issues for a moment and, as Mankiw-Romer-Weil did, let us proxy s_h with the fraction of working-age population that is in school (and let us also set $\delta + x = .05$). Also, we will measure s_k with the investment-to-GDP ratio and Y/L with GDP per worker.

- Now, let us first run the MRW regression with the restriction that $\beta = 0$ (i.e., no human capital):

$$\ln\left(\frac{Y}{L}\right)_{j,t} = \text{constant} + b_1 \ln\left(s_{k,j}\right) + b_2 \ln\left(n_j + x + \delta\right) + a_j.$$

Then the results are as in Table 1 (reproduced from Acemoglu 2007).

Table 1: MRW for the basic Solow Model

	MRW	Updated data	
	1985	1985	2000
$\ln(s_{kj})$	1.42	1.01	1.22
	(.14)	(.11)	(.13)
$\ln(n_j + x + \delta)$	-1.97	-1.12	-1.31
	(.56)	(.55)	(.36)
Adj R^2	.59	.49	.49
Implied α	.59	.50	.55
No. of observations	98	98	107

- Note that cross-country variation in investment rates s_k and population growth rates n can account for a bit more than 50% of the observed cross-country variation in productivity levels.

- However, this is acheived with α close to 0.6, which is inconsistent with the income share of capital observed in the data (α between 0.3 and 0.4). If we were to impose $\alpha = .35$ ($b_1 = -b_2 = .54$), and see how the calibrated Solow model fits the data, then cross-country differences in s_k and n would account for only 1/4 to 1/3 of the observed cross-country productivity differences. (This is half than before, but still it's a pretty good fit.)

- But, what if we use the augmented Solow model? Then the results look better, as in Table 2.

Table 2. MRW for augmented Solow model

	MRW	Updated data	
	1985	1985	2000
$\ln(s_k)$	.69	.65	.96
	(.13)	(.11)	(.13)
$\ln(s_h)$	.66	.47	.70
	(.07)	(.07)	(.13)
$\ln(n+g+\delta)$	-1.73	-1.02	-1.06
	(.41)	(.45)	(.33)
Adj R^2	.78	.65	.60
Implied α	.30	.31	.36
Implied β	.28	.22	.26
No. of observations	98	98	107

- Therefore, the augmented Solow model can account for 60% to 80% of the observed cross-country differences in productivity levels, with an α that is comfortably within the range of empirically plausible values.

- However, there are important caveats.

- First, if one uses alternative measures for human capital, such as secondary-eductation attainment levels, then the fit is significantly reduced (Klenow and Rodriguez, 1997).

- Second, it is not clear how one should interpret any of these results, given that there are good reasons to expect (as we will see in the models that follow) that technological differences are likely to be correlated with saving rates and education levels—indeed with the causality probably going both ways.

- Third, whereas the estimated α is empirically plausible, the estimated β is not. In particular, the β estimated in MRW implies much higher returns to education than the ones estimated with standard Mincerian regressions on micro data (Klenow and Rodrigez, 1997; Hall and Jones, 1999). If one calibrates the augmented Solow model so that β is consistent with Mincerian estimates, then the model accounts for about one half of the observed cross-country output-per-worker differences—which is leaves the rest one half "explained" by technological difference (the augemented-Solow residuals a_j).

- If we calibrate the augmented Solow model as in Hall and Jone (1999), use it to predicted the output-per-worker levels for each country, and and plot them against the actual levels of each country, then we get Figure 2.5 (reproduced from Acemoglu 2007). Clearly, there is significant unexplained variation. Moreover, the "errors" are strongly correlated with actual levels—the model particularly overestimates the output-per-worker levels of poor countries.

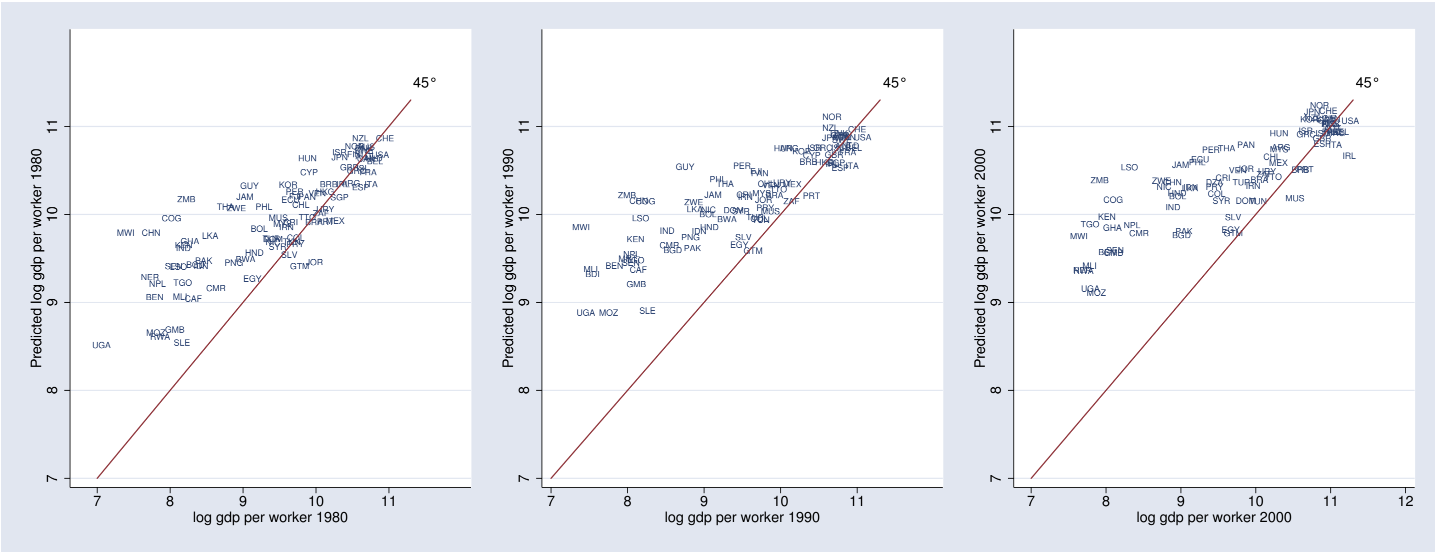

Figure 2.5: Calibrated GDP per worker from augmented Solow model versus actual GDP per worker (in logs).

2.6 Conditional Convergence: "Barro" regressions

- Recall the log-linearization of the dynamics around the steady state:

$$\frac{\dot{y}}{y} = \lambda \ln \frac{y}{y^*}.$$

A similar relation will hold true in the neoclassical growth model a la Ramsey-Cass-Koopmans. $\lambda < 0$ reflects local diminishing returns. Such local diminishing returns occur even in endogenous-growth models. The above thus extends well beyond the simple Solow model.

- Rewrite the above as

$$\Delta \ln y = \lambda \ln y - \lambda \ln y^*$$

Next, let us proxy the steady state output by a set of country-specific controls X, which include s, δ, n, τ etc. That is, let

$$-\lambda \ln y^* \approx \beta' X.$$

We conclude

$$\Delta \ln y = \lambda \ln y + \beta' X + error$$

- The above represents a typical "Barro-like" conditional-convergence regression: We use cross-country data to estimate λ (the convergence rate), together with β (the effects of the saving rate, education, population growth, policies, etc.) The estimated convergence rate is about 2% per year. The estimated coefficients on X are of the expected sign.

- These types of regressions are quite informative: they identify important correlations in the data. However, inferring causality is much trickier. (See Barro and Sala-i-Martin's and Acemoglu's textbooks for further discussion.)

2.7 The Golden Rule

- *The Golden Rule.* Consumption at the steady state is given by

$$c^* = (1-s)f(k^*) = f(k^*) - (\delta + n)k^*$$

Suppose society chooses s so as to maximize c^*. Since k^* is a monotonic function of s, this is equivalent to choosing k^* so as to maximize c^*. Note that

$$c^* = f(k^*) - (\delta + n)k^*$$

is strictly concave in k^*. The FOC is thus both necessary and sufficient. c^* is thus maximized if and only if $k^* = k_{gold}$, where k_{gold} solves

$$f'(k_{gold}) - \delta = n.$$

Equivalently, $s = s_{gold}$, where s_{gold} solves $s_{gold} \cdot \phi(k_{gold}) = (\delta + n)$ This is called the "*golden rule* " for savings, after Phelps.

- *Dynamic Inefficiency.* If $s > s_{gold}$ (equivalently, $k^* > k_{gold}$), the economy is dynamically inefficient: if the saving rate is lowered to $s = s_{gold}$ for all t, then consumption in all periods will be higher! On the other hand, if $s < s_{gold}$ (equivalently, $k^* > k_{gold}$), then raising s towards s_{gold} will increase consumption in the long run, but at the expense of lower consumption in the short run; whether such a trade-off is desirable depends on how one weighs current generations vis-a-vis future generations.

- *The Modified Golden Rule.* In the Ramsey model, this trade-off will be resolved when k^* satisfies the

$$f'(k^*) - \delta = n + \rho,$$

where $\rho > 0$ measures impatience, or discounting across generations. Clearly, the distance between the Ramsey-optimal k^* and the golden-rule k_{gold} increases with ρ.

- *Testing for dynamic inefficiency.* The golden rule can be restated as $r - \delta = \dot{Y}/Y$; dynamic efficiency is ensured if $r - \delta > \dot{Y}/Y$. Abel et al. use test this relation with US data and find no evidence of dynamic inefficiency.

2.8 Poverty Traps, Cycles, Endogenous Growth, etc.

- The assumptions we have imposed on savings and technology implied that G is increasing and concave, so that there is a unique and globally stable steady state. More generally, however, G could be non-concave or even non-monotonic. Such "pathologies" can arise, for example, when the technology is non-convex, as in the case of locally increasing returns, or when saving rates are highly sensitive to the level of output, as in some OLG models.

- Figure 2.6 illustrates an example of a non-concave G. There are now *multiple* steady states. The two extreme ones are (locally) stable, the intermediate is unstable versus stable ones. The lower of the stable steady states represents a *poverty trap*.

- Figure 2.7 illustrates an example of a non-monotonic G. We can now have oscillating dynamics, or even perpetual endogenous cycles.

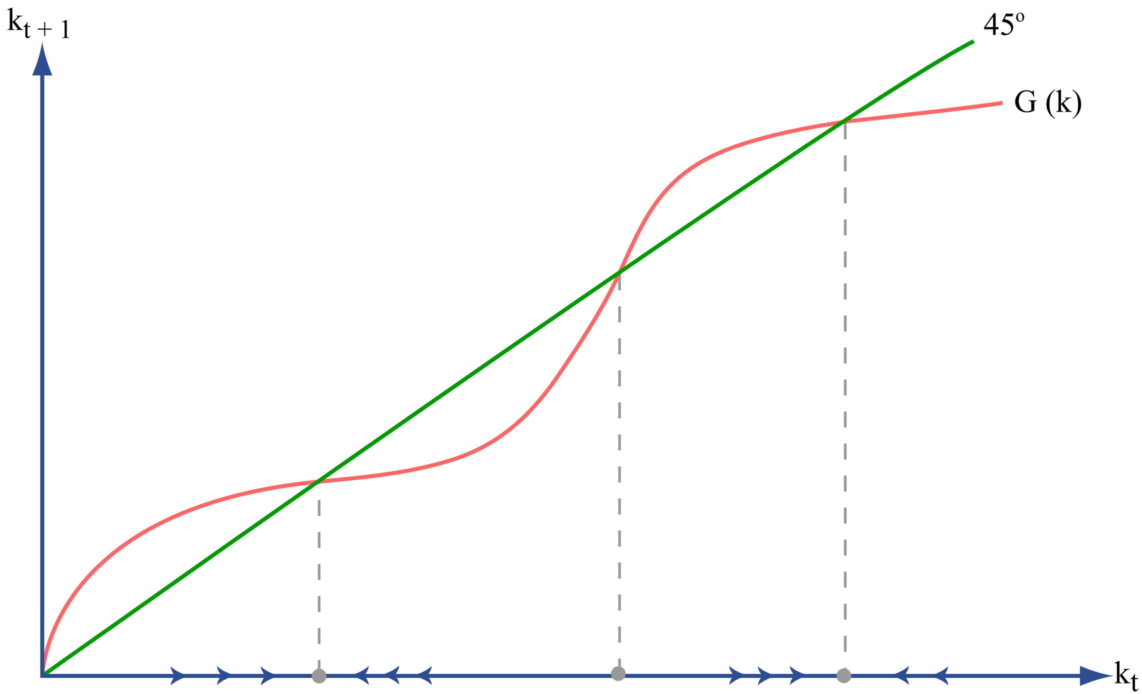

Figure 2.6: Non-concave G and poverty traps.

Figure by MITOCW.

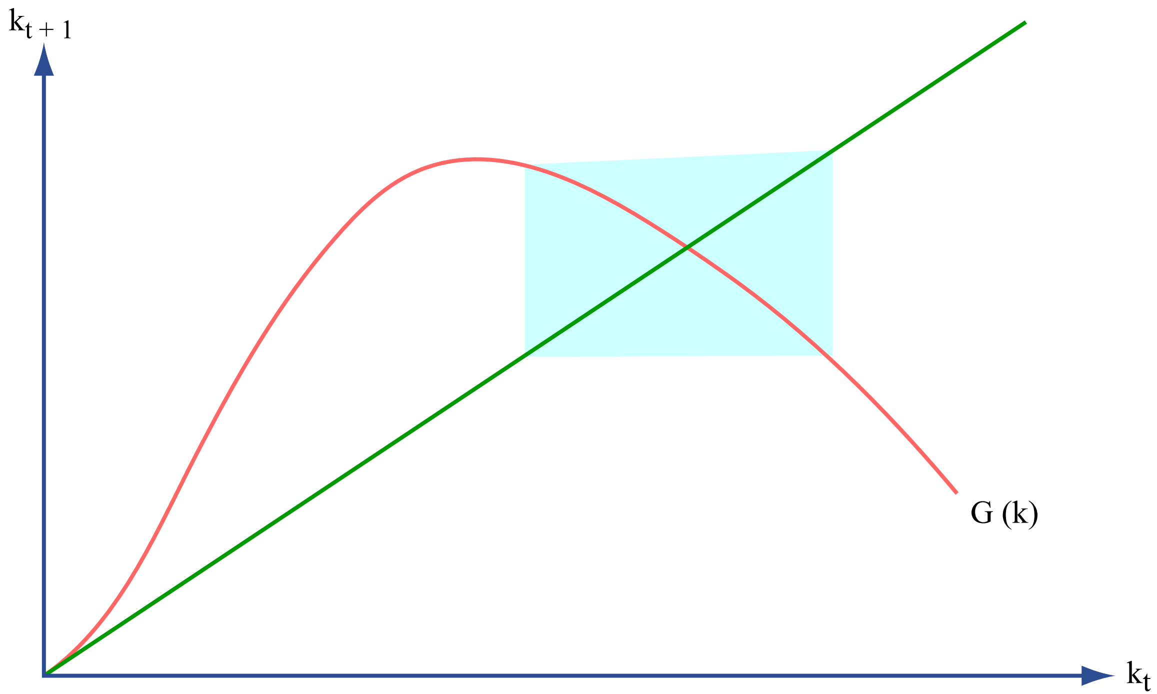

Figure 2.7: Non-monotonic G and cycles.

Figure by MIT OCW.

- What ensures that the growth rate asymptotes to zero in the Solow model (and the Ramsey model as well) is the vanishing marginal product of capital, that is, the Inada condition $\lim_{k\to\infty} f'(k) = 0$.

- Continue to assume that $f''(k) < 0$, so that $\gamma'(k) < 0$, but assume now that $\lim_{k\to\infty} f'(k) = A > 0$. This implies also $\lim_{k\to\infty} \phi(k) = A$. Then, as $k \to \infty$,

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} \to sA - (n + \delta)$$

- If $sA < (n + \delta)$, then it is like before: The economy converges to k^* such that $\gamma(k^*) = 0$. But if $sA > (n + x + \delta)$, then the economy exhibits deminishing but not vanishing growth: γ_t falls with t, but $\gamma_t \to sA - (n + \delta) > 0$ as $t \to \infty$.

- Jones and Manuelli consider such a general convex technology: e.g., $f(k) = Bk^\alpha + Ak$. We then get both transitional dynamics in the short run and perpetual growth in the long run.

- In case that $f(k) = Ak$, the economy follows a balanced-growth path from the very beginning.

- We will later "endogenize" A in terms of externalities, R&D, policies, institutions, markets, etc.